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A Short Course on Extreme Value Statistics in Applications

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The Bivariate ACER Method

Bivariate Extreme Value Distributions

Consider a bivariate stochastic process $Z(t) = \big(X(t), Y(t)\big)$ with dependent component processes, which has been observed over a time interval, $(0, T)$ say.

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Assume that the sampled values $(X_1, Y_1), \dots, (X_N, Y_N)$ are allocated to the (usually equidistant) discrete times $t_1, \dots, t_N$ in $(0, T)$.

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Our goal is to accurately determine empirically the joint distribution function of the extreme value vector $\big(M_{x,N}, M_{y,N}\big)$, where $M_{x,N} = \max\big\{X_j\, ; j = 1, \dots, N\big\}$, and with a similar definition of $M_{y,N}$.

Specifically, we want to estimate $P(\xi, \eta) = \text{Prob}\,\big(M_{x,N} \le \xi, M_{y,N} \le \eta\big)$ accurately for large values of ξ and η.

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Bivariate Extreme Value Distributions

By transformation of variables, any other marginal distribution can be obtained from the standard Fréchet distribution $F(z) = \exp(-1/z)$, $z > 0$. This is a special case of the GEV distribution with parameters $\mu = 0$, $\sigma = 1$ and $\gamma = 1$.

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Now, it follows that $\text{Prob}(M_{x,N} \leq z) = \exp(-N/z)$, or, equivalently, $\text{Prob}(M_{x,N}/N \leq z) = \exp(-1/z)$, $z > 0$. The same result applies to $M_{y,N}$.

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Now, it follows that $\text{Prob}(M_{x,N} \leq z) = \exp(-N/z)$, or, equivalently, $\text{Prob}(M_{x,N}/N \leq z) = \exp(-1/z)$, $z > 0$. The same result applies to $M_{y,N}$.
Hence, to obtain standard univariate results for each margin, we should consider the re-scaled vector,

$$M_N^* = (M_{x,N}^*, M_{y,N}^*) = (M_{x,N}/N, M_{y,N}/N).$$

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Bivariate Extreme Value Distributions

Then if,

$$\text{Prob}(M^*_{x,N} \leq x, M^*_{y,N} \leq y) \rightarrow G(x,y), \text{ as } N \rightarrow \infty,$$

where G is a non-degenerate distribution function, G has the form,

$$G(x,y) = \exp\{-V(x,y)\}, \;\; x > 0, \;\; y > 0,$$

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Here

$$V(x,y) = \int_0^1 2 \ \max\left(\frac{w}{x}, \frac{1-w}{y}\right) dH(w),$$

where H is a distribution function on $[0,1]$ satisfying the mean value constraint

$$\int_0^1 w \, dH(w) = 1/2.$$

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Bivariate Extreme Value Distributions

For any GEV marginal, it is only necessary to transform the marginals from standard Fréchet to the required members of the GEV family. Specifically, by defining,

$$\tilde{x} = \left\{1 + \gamma_x\left(\frac{x - \mu_x}{\sigma_x}\right)\right\}^{1/\gamma_x} \quad \text{and} \quad \tilde{y} = \left\{1 + \gamma_y\left(\frac{y - \mu_y}{\sigma_y}\right)\right\}^{1/\gamma_y},$$

it follows that the complete set of bivariate asymptotic extreme value distributions is determined by distribution functions of the form,

$$G(x, y) = \exp\{-V(\tilde{x}, \tilde{y})\},$$

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Cascade of Approximations

We introduce the non-exceedance event $\mathcal{C}_{kj}(\xi,\eta) = \{X_{j-1} \le \xi, Y_{j-1} \le \eta, \ldots, X_{j-k+1} \le \xi, Y_{j-k+1} \le \eta\}$ for $1 \le k \le j \le N+1$. Then, from the definition of $P(\xi,\eta)$ it follows that,

$$
\begin{aligned}
P(\xi,\eta) &= \text{Prob}\big(\mathcal{C}_{N+1,N+1}(\xi,\eta)\big) \\
&= \text{Prob}\big(X_N \le \xi, Y_N \le \eta \,|\, \mathcal{C}_{NN}(\xi,\eta)\big) \cdot \text{Prob}\big(\mathcal{C}_{NN}(\xi,\eta)\big) \\
&= \prod_{j=2}^{N} \text{Prob}\big(X_j \le \xi, Y_j \le \eta \,|\, \mathcal{C}_{jj}(\xi,\eta)\big) \cdot \text{Prob}\big(\mathcal{C}_{22}(\xi,\eta)\big).
\end{aligned}
$$

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Cascade of Approximations

The following representation applies for a suitably chosen k,

$$P(\xi,\eta) \approx \exp\Big\{ -\sum_{j=k}^{N} \big(\alpha_{kj}(\xi;\eta) + \beta_{kj}(\eta;\xi) - \gamma_{kj}(\xi,\eta)\big)\Big\}; \;\; \xi,\eta \to \infty\,,$$

where we have used the notation
$\alpha_{kj}(\xi;\eta) = \text{Prob}\big(X_j > \xi \,|\, \mathcal{C}_{kj}(\xi,\eta)\big)$,
$\beta_{kj}(\eta;\xi) = \text{Prob}\big(Y_j > \eta \,|\, \mathcal{C}_{kj}(\xi,\eta)\big)$ and
$\gamma_{kj}(\xi,\eta) = \text{Prob}\big(X_j > \xi,\, Y_j > \eta \,|\, \mathcal{C}_{kj}(\xi,\eta)\big)$.

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Cascade of Approximations

The k'th order bivariate ACER function is given by,

$$\mathcal{E}_k(\xi,\eta) = \frac{1}{N-k+1} \sum_{j=k}^{N} \big(\alpha_{kj}(\xi;\eta) + \beta_{kj}(\eta;\xi) - \gamma_{kj}(\xi,\eta)\big); \quad k = 1, 2, .$$

Hence, when $N \gg k$, we may write

$$P(\xi,\eta) \approx \exp\{-(N-k+1)\,\mathcal{E}_k(\xi,\eta)\} \; ; \; \xi,\eta \to \infty \, .$$

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Bivariate Extreme Value Copulas

The Gumbel logistic:

$$\mathcal{G}_k(\xi, \eta) = \left[\left(\varepsilon_k^x(\xi) \right)^m + \left(\varepsilon_k^y(\eta) \right)^m \right]^{\frac{1}{m}}.$$

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The Asymmetric logistic:

$$\mathcal{A}_k(\xi,\eta) = \left[\left(\phi\varepsilon_k^x(\xi)\right)^m + \left(\theta\varepsilon_k^y(\eta)\right)^m\right]^{\frac{1}{m}} + (1-\phi)\varepsilon_k^x(\xi) + (1-\theta)\varepsilon_k^y(\eta).$$

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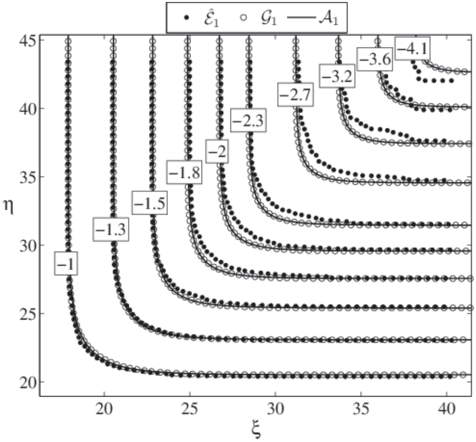
Bivariate Extreme Value Copulas



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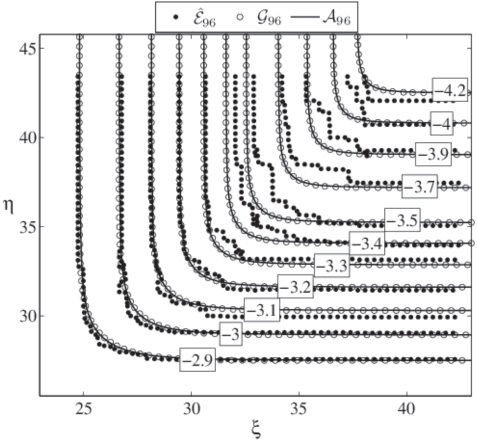

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