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Fast and accurate pricing of discretely monitored barrier options by numerical path integration

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Abstract Barrier options are financial derivative contracts that are activated or deactivated according to the crossing of specified barriers by an underlying asset price. Exact models for pricing barrier options assume continuous monitoring of the underlying dynamics, usually a stock price. Barrier options in traded markets, however, nearly always assume less frequent observation, e.g. daily or weekly. These situations require approximate solutions to the pricing problem. We present a new approach to pricing such discretely monitored barrier options that may be applied in many realistic situations. In particular, we study daily monitored up-and-out call options of the European type with a single underlying stock. The approach is based on numerical approximation of the transition probability density associated with the stochastic differential equation describing the stock price dynamics, and provides accurate results in less than one second whenever a contract expires in a year or less. The flexibility of the method permits more complex underlying dynamics than the Black and Scholes paradigm, and its relative simplicity renders it quite easy to implement.

Keywords Barrier options · Discrete monitoring · Numerical path integration

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1 Introduction

An option or a financial derivative security is an agreement between two contractual partners that gives the holder of the option the right but not the obligation to buy (call) or sell (put) another asset some time in the future at an agreed-upon price. Often the underlying asset is a stock, but options may depend on the value of almost any other traded asset, like real estate or metals.

We are concerned with barrier options, i.e. options where the holder gains or loses his right to buy or sell when the price of the underlying stock crosses a barrier specified by the contract. Many different contracts can be imagined. If a barrier is set below the initial stock price, the contract may render the option worthless if the stock price falls below the barrier (down-and-out option). Another possibility is that the holder can not exercise his right unless the stock price exceeds a certain barrier (up-and-in option). Combinations of barriers and time-dependence are also possible.

Option pricing is based on a mathematical model of the underlying stock price dynamics. The most popular is the Black and Scholes (1973) model, which assumes that the stock price develops according to a geometric Brownian motion. An advantage of this assumption is the possibility of deriving closed-form solutions to the pricing problem. This model has its limitations, but continues to be useful also as a building block for more complicated models.

Exact pricing of barrier options in a Black–Scholes market is possible as long as the holder can exercise his right at any time (see e.g. Hull (2003)). But in traded markets this right is usually limited to discrete monitoring times, e.g. once a day or once a week. It is therefore necessary to seek approximate solutions to the pricing problem. A number of strategies have been suggested, including binomial trees, trinomial trees, finite difference methods, finite element methods and Monte Carlo simulation, apart from analytical methods. However, none of these appear to combine high accuracy, computational efficiency and general applicability.

We will present a conceptually simple, general and easily implementable method that is based on numerically integrating the transition probability densities of the stochastic differential equation. Although we will test the method on an example that assumes geometric Brownian motion as the underlying dynamics, this is by no means required by the method.

2 Model

Assume a market where the risk-adjusted stock price develops according to the stochastic differential equation

$$dS_t = \mu(S_t)\,dt + \sigma(S_t)\,dW_t, \tag{1}$$

where W_t is a standard Brownian motion. Initially, no restrictions are placed on the drift function $\mu(s)$ or the diffusion function $\sigma(s)$, apart from the regularity conditions required for Eq. 1 to be well-defined (see e.g. Øksendal, 2003), and such that the process S_t is characterized by an absolutely continuous probability distribution.

We shall be concerned with an up-and-out call option of the European type written on a single stock having a constant barrier B and a strike price X in a market with a risk-free interest rate r. The initial stock price is equal to $S_0 = s_0$. The option becomes worthless as soon as the stock price S_t is greater than B at one of the discrete monitoring times. Of course, $s_0 < B$ and $X < B$. If a price greater than B is not observed, the option's value at maturity T is $\max(0, S_T - X)$. The underlying stock is monitored at m different times $\tau_j, j = 1, \ldots, m$ until maturity such that $0 < \tau_1 < \tau_2 < \ldots < \tau_{m-1} < \tau_m = T$. We thus observe a time series of stock prices $S_0, S_{\tau_1}, \ldots, S_{\tau_m}$. To calculate the option's value we consider the barrier process

$$\tilde{S}_t = S_t I \left[S_{\tau_i} \leq B \text{ for every} \tau_i, 0 < \tau_i \leq t \right], \tag{2}$$

which is then defined as the price process at time t multiplied with the indicator function of the event that the observed price process has not exceeded the barrier up to time t, where $I[A] = 1$ if the event A has occurred, and $I[A] = 0$ otherwise. Now define a probability function

$$H_m(s) = P\{\tilde{S}_{\tau_m} > s\} = P\{s < S_{\tau_m} \leq B \cap S_{\tau_j} \leq B; 1 \leq j < m\}, \tag{3}$$

for $0 < s < B$. Then $g_m(s) = -\mathrm{d}H_m(s)/\mathrm{d}s$ is given by

$$g_m(s) = \int_0^B \ldots \int_0^B p_{m|m-1}(s|s_{m-1}) \ldots p(s_2|s_1) p(s_1|s_0) ds_1 \ldots ds_{m-1}, \tag{4}$$

where $p_{i|i-1}(s|s')$ denotes the transition probability density function of S_t from τ_{i-1} to τ_i. The option price is then equal to

$$p = \exp(-rT) \int_X^B (s - X) g_m(s) \mathrm{d}s, \tag{5}$$

when the interest rate r is constant over the maturity time T. Note that as B goes to infinity, $g_m(s)$ approaches the PDF of the risk-free price process at maturity, which can be used to price a plain vanilla option.

3 Implementation

Whenever the transition probability density $p_{i|i-1}(s|s'), i = 1, \ldots, m$, is at hand, Eq. 4 can be used recursively to obtain $g_m(s)$:

$$g_2(s) = \int_0^B p_{2|1}(s|s') p_{1|0}(s'|s_0) \mathrm{d}s', \tag{6}$$

and

$$g_i(s) = \int_0^B p_{i|i-1}(s|s')g_{i-1}(s')ds', i = 3, \ldots, m. \tag{7}$$

In practice, each element in the series of functions

$$p_{1|0}(s|s_0), g_2(s), g_3(s), \ldots, g_{m-1}(s), g_m(s)$$

is represented on a numerical grid. In order to save memory and computing time, the modest number of 80 uniformly spaced grid points are placed along the intervals where these functions are expected to have essentially non-zero values.

A limited number (1000) of Monte Carlo simulations of S_t are carried out in order to find reasonable upper and lower limits for the grid points s_i. Under any circumstances, the upper cut-off value is never set greater than B.

Under the Black and Scholes framework

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma S_t\mathrm{d}W_t, \tag{8}$$

where σ is the constant stock volatility, the transition probability density is explicitly given as

$$p_{i+1|i}(s|s') = \frac{1}{\sqrt{2\pi\Delta\tau_i}\sigma s}\exp\left(-\frac{\left(\log(s) - \log(s') - (r - \sigma^2/2)\Delta\tau_i\right)^2}{2\sigma^2\Delta\tau_i}\right), \tag{9}$$

where $\Delta\tau_i = \tau_{i+1} - \tau_i$. Hence, whatever the length of the observation intervals $\Delta\tau_i$, the accuracy of the recursive scheme above for the Black and Scholes model will depend solely on the accuracy of the numerical integrations carried out.

If the exact transition probability density is not available, one possibility is to use an approximate density derived from an Euler-Maruyama discretization (see e.g. Kloeden & Platen, 1992) of the stochastic process S_t, which is obtained from Eq. 1:

$$S_{\tau_{i+1}} = S_{\tau_i} + \mu(S_{\tau_i})\Delta\tau_i + \sigma(S_{\tau_i})\Delta W_{\tau_i}, \tag{10}$$

where $\Delta W_{\tau_i} = W_{\tau_{i+1}} - W_{\tau_i}$ is normally distributed with expectation zero and variance equal to $\Delta\tau_i$. The approximate transition probability density then becomes

$$p_{i+1|i}(s|s') = \phi_N(s; s' + \mu(s')\Delta\tau_i, \sigma^2(s')\Delta\tau_i), \tag{11}$$

where

$$\phi_N(s;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(s-\mu)^2}{2\sigma^2}\right). \tag{12}$$

However, in general the observation interval is too long for this approximation to be very accurate. A possible remedy is to invoke the Markov property of S_t, which allows us to express the transition probability density $p_{i+1|i}(s|s')$ as follows: Let $\tau_i = t_0^{(i)} < t_1^{(i)} < \cdots < t_{n_i}^{(i)} = \tau_{i+1}$, then, by the Chapman-Kolmogorov equation, $(x_0 = s', x_{n_i} = s)$

$$p_{i+1|i}(s|s') = \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^{n_i} p_{t_j^{(i)}|t_{j-1}^{(i)}}(x_j|x_{j-1})\mathrm{d}x_1 \ldots \mathrm{d}x_{n-1} \tag{13}$$

where the transition probability density functions $p_{t_j^{(i)}|t_{j-1}^{(i)}}(x_j|x_{j-1})$ are again given by Eq. 11 where $\Delta\tau_i$ is replaced by $\Delta t_j^{(i)} = t_{j+1}^{(i)} - t_j^{(i)}$. Clearly the accuracy of Eq. 13 depends on the quantity $\max\{\Delta t_j^{(i)}; j = 1, \ldots, n_i\}$.

The complete recursion algorithm may now be written as follows: Let the obtained total discretization be written as $0 < t_1 < \ldots < t_n = T$, where $n = \sum_{i=1}^{m-1} n_i$. Let $b(j) = B$ if $t_j = \tau_i$ for some $i \in \{1, \ldots, m\}$, else $b(j) = \infty$, and let $\tilde{g}_j(\cdot), j = 2, \ldots, n$ be defined as

$$\tilde{g}_2(s) = \int_0^{b(t_1)} p_{t_2|t_1}(s|x)p_{t_1|0}(x|s_0)\mathrm{d}x, \tag{14}$$

and

$$\tilde{g}_j(s) = \int_0^{b(t_{j-1})} p_{t_j|t_{j-1}}(s|x)\tilde{g}_{j-1}(x)\mathrm{d}x, \quad j = 3, \ldots, n, \tag{15}$$

where then finally $\tilde{g}_n(s) = g_m(s)$ for $0 < s < B$, within the approximation of the given discretization.

So far the analysis has been based on the Euler-Maruyama approximation to Eq. 1, which centers on the approximation

$$\int_{t_j}^{t_{j+1}} \sigma(S_t)\,\mathrm{d}W_t = \sigma\left(S_{t_j}\right)(W_{t_{j+1}} - W_{t_j}) = \sigma\left(S_{t_j}\right)\Delta W_{t_j}. \tag{16}$$

The advantage of this approximation is obvious from the preceding analysis, viz. that the transition probability density $p_{t_{j+1}|t_j}(s|s')$ can be represented by a Gaussian density. As discussed extensively in Kloeden & Platen (1992), there are several ways of improving on the simple Euler-Maruyama approximation, both by weak and strong discretization schemes. Since the goal here is to calculate the probability density functions $\tilde{g}_j(s)$ of the option price process, it is sufficient to limit the attention to the weak schemes. In particular, the simplified weak Taylor scheme of order 2.0 will be discussed. It may be noted that the Euler-Maruyama scheme is of weak order 1.0. According to Kloeden & Platen (1992), the simplified weak order 2.0 Taylor scheme for the conditional random variable $\tilde{S}_{j+1} = \{S_{t_{j+1}} | S_{t_j} = s_j\}$ may be written as

$$\tilde{S}_{j+1} = \alpha_j + \beta_j \Delta W_{t_j} + \gamma_j \Delta W_{t_j}^2. \tag{17}$$

Here

$$\begin{aligned}\alpha_j = s_j + \mu(s_j)\Delta t_j - \sigma(s_j)\sigma'(s_j)\Delta t_j/2 \\ + \left(\mu(s_j)\mu'(s_j) + \mu''(s_j)\sigma(s_j)^2/2\right) \Delta t_j^2/2,\end{aligned} \tag{18}$$

$$\beta_j = \sigma(s_j) + \left(\mu'(s_j)\sigma(s_j) + \mu(s_j)\sigma'(s_j) + \sigma''(s_j)\sigma(s_j)^2/2\right) \Delta t_j/2, \tag{19}$$

and

$$\gamma_j = \sigma(s_j)\sigma'(s_j)/2. \tag{20}$$

The prime $'$ denotes differentiation, that is, $\mu'(s) = \mathrm{d}\mu(s)/\mathrm{d}s$, and so on. Convergence of the present weak Taylor scheme of order 2.0 is guaranteed if the functions $\mu(s)$ and $\sigma(s)$ satisfy certain regularity conditions, cf. Kloeden & Platen (1992).

Having achieved the representation of Eq. 17, we may proceed to calculate $p_{t_{j+1}|t_j}(s|s_j)$. This transition probability density can, of course, still be expressed in closed form since $\tilde{S}_{j+1}$ is a quadratic expression in the Gaussian variable ΔW_{t_j}. Let $\xi_j^{\pm}$ denote the two solutions of the equation

$$s = h(\xi) = \alpha_j + \beta_j \xi + \gamma_j \xi^2. \tag{21}$$

That is

$$\xi_j^{\pm} = -\beta_j/(2\gamma_j) \pm \sqrt{(s-\alpha_j)/\gamma_j + (\beta_j/(2\gamma_j))^2}. \tag{22}$$

It is then obtained that

$$\begin{aligned}p_{t_{j+1}|t_j}(s|s_j) &= \sum_{\varepsilon=+,-} \frac{\phi_N(\xi_j^{\varepsilon}; 0, \Delta t_j)}{|h'(\xi_j^{\varepsilon})|} \\ &= \frac{\phi_N(\xi_j^{+}; 0, \Delta t_j) + \phi_N(\xi_j^{-}; 0, \Delta t_j)}{\sqrt{(s-\alpha_j)/\gamma_j + (\beta_j/(2\gamma_j))^2}}\end{aligned} \tag{23}$$

for $(s-\alpha_j)/\gamma_j + (\beta_j/(2\gamma_j))^2 > 0$. So even if the transition probability density $p_{t_{j+1}|t_j}(s|s_j)$ is more complicated for the weak order 2.0 approximation above than the previous transition probability density, which was simply a Gaussian density, it is still tractable for numerical calculations. It is therefore of interest to explore the impact of this approximation on the numerical accuracy of the calculated values for the option price by combining Eqs. 6 and 7 or 14 and 15 with Eq. 23.

To carry out the numerical integrations, we use Simpson's method with 400 partitions. When the algorithm calls for a value of g_{i-1} or $\tilde{g}_{j-1}$ outside the chosen grid, the value of a cubically interpolated spline is provided. The integrations in Eqs. 6 and 7 or 14 and 15 are limited to the intervals where almost all the narrow transition density's mass is localized. Given s, we limit the integration to the interval defined by a backwards Euler-Maruyama step plus–minus six standard deviations as determined by the transition density.

4 Numerical results

We base our numerical experiments on an up-and-out call option in a Black and Scholes market with initial stock price 110, strike 100, interest rate 10%, volatility 30% and time to maturity 0.2 years, i.e. 50 trading days. Barriers are in intervals of five between 115 and 155 and the stock price is monitored daily. The situation has been studied by Broadie et al. (1997), who derived an analytical approximation to the pricing problem. For comparison they calculated the true values using a trinomial tree with 80,000 partitions. This approach is computationally expensive, but their results will serve us the same purpose.

To try out our path integral approach, we have tested the method using different implementations:

1. Exact transition probability density
2. Taylor based transition probability density with 1-day discretization intervals
3. Euler-Maruyama based transition probability density dividing each day into 5 discretization intervals, that is, $n_i = 5$ in Eq. 13.
4. Euler-Maruyama based transition probability density with 1-day discretization intervals

The CPU time is a fraction of a second for the implementations 1, 2, and 4, and slightly above a second for number 3. As we can see in the Table 1, the exact and the Taylor based transition density come up with identical results that are practically equal to the benchmark results. Whenever the underlying stock price dynamics are more complicated and the exact transition density is not available, PI 2 should still provide good results. With the Euler-Maruyama based transition density some accuracy is lost. But is compares rather well with e.g. the approximate results of Kou (2003), who has studied the same example, and it is improved by dividing the days in 5 intervals. The disadvantage of this finer discretization is a fivefold increase in the CPU time. It must also be noted that further discretization does not produce more accurate results without refining the grid, which will increase the CPU time even more.

Table 1 Option price results

B	True	PI 1	PI 2	PI 3	PI 4	Kou	Duan et al.
115	0.807	0.806	0.806	0.806	0.806	0.819	0.807
120	2.418	2.418	2.418	2.416	2.419	2.442	
125	4.616	4.616	4.616	4.615	4.623	4.649	
130	6.922	6.922	6.922	6.922	6.936	6.959	
135	8.959	8.959	8.959	8.960	8.979	8.994	8.958
140	10.551	10.552	10.552	10.553	10.574	10.581	
145	11.684	11.685	11.685	11.686	11.706	11.707	
150	12.431	12.432	12.432	12.434	12.451	12.448	
155	12.894	12.895	12.895	12.897	12.905	12.894	12.894

Duan et al. (2003) also provide very accurate results in a short time with a method that is similar to ours in the sense that is exploits the Markov property of the stochastic differential equation, but our method is perhaps conceptually simpler and consequently very easy to program. Not counting library routines for interpolation and an external random number generator, the program written to perform our calculations consists of about 100 lines of FORTRAN code.

5 Accuracy and computational cost

Broadie et al. demonstrated that reasonably accurate pricing could be obtained by applying the continuous barrier formula after slightly moving the barrier. This analytical solution to the pricing problem permits nearly instant pricing of a large number of barrier options (1000s in a second). Using numerical path integration we have priced one option in 0.2 seconds when time to maturity is 50 days (the price corresponding to a different strike can be found at an insignificant computational cost as long as the function g in (5) has been found). Although clever implementations may reduce the CPU time, is is clear that path integration can never compete with an analytical approximation as far as speed is concerned. On the other hand it provides more accurate results, particularly when the barrier is close to the initial price, as can be seen in Table 2. And it remains orders of magnitude faster than using the trinomial tree. The choice of method thus depends on the trade-off between computational speed and accuracy.

6 Conclusion

We have demonstrated that the price of a European call option in a Black–Scholes market with an up-and-out barrier that is monitored daily for 50 days can be estimated very accurately in a fraction of a second by recursive numerical integration of the transition probability density associated with the stochastic differential equation describing the risk-adjusted stock price dynamics. To take full advantage of the potential of this path integration scheme it is crucial how

Table 2 Comparison to corrected continuous barrier method

B	True	Broadie et al.	Err. (%)	PI 2	Err. (%)
115	0.807	0.819	1.5	0.806	0.124
120	2.418	2.442	1.0	2.418	0.000
125	4.616	4.649	0.7	4.616	0.000
130	6.922	6.959	0.5	6.922	0.000
135	8.959	8.994	0.4	8.959	0.000
140	10.551	10.581	0.3	10.552	0.009
145	11.684	11.707	0.2	11.685	0.009
150	12.431	12.448	0.1	12.432	0.008
155	12.894	12.905	0.1	12.895	0.008

the numerical scheme is implemented. Here we have described briefly a few such implementation strategies.

The same method might as well have been applied to many other cases, including up-and-in, down-and-in, down-and-out or double barrier options. The essential requirement is that the price can be found by repeated calculation of an integral over transition densities, like Eq. 4. It is clear that by changing the upper integration limits of 4, we might also have priced an option with a time-varying barrier. Since the integral can be suited to the monitoring frequency of the option, this approach is uniquely flexible.

Finally, it should also be noted that the Euler-Maruyama or Taylor approximations can be used to estimate the price of options with more complex underlying dynamics by essentially the same method. The path integration method in combination with a suitable approximation scheme for the transition probability density and an interpolation procedure, provided very accurate results in the classical Black–Scholes case. On the basis of a consideration of each of the elements entering the numerical solution procedure, it appears possible to accurately price a wide variety of options in this way as long as the underlying dynamics is driven by a process with stationary, independent increments, whether it be Brownian motion or not. This generality of the method suggests numerous possibilities of future developments.

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