Norges teknisk–
naturvitenskapelige universitet
Institutt for matematiske fag

English

Contact during exam:
Arvid Næss 73 59 70 53/ 99 53 83 50

EXAM IN COURSE TMA4265 STOCHASTIC PROCESSES

Tuesday, December 13, 2011
Time: 9:00–13:00

Permitted aid items:

- Yellow A-5 sheet with your own handwritten notes (stamped by the Department of Mathematical Sciences)
- *Tabeller og formler i statistikk*, Tapir Forlag
- K. Rottmann: *Matematisk formelsamling*
- Calculator HP30S

The results from the exam are due by January 13, 2012.

Problem 1 - ON/OFF System

Consider a system that alternates between the two states 0 (OFF) and 1 (ON), and assume that it is checked at discrete timepoints $1, 2, \ldots$. If the system is OFF at one timepoint, the probability that it has switched to ON at the next timepoint is p; and if it is ON, the probability that it has switched to OFF is q. Here $0 \leq p, q \leq 1$.

a) Describe the system as a Markov chain X_n, $n = 0, 1, 2, \ldots$, and establish the transition probability matrix $\mathbf{P}$. Show that for $m = 1, 2, \ldots$,

$$\mathbf{P}^m = \frac{1}{p+q}\begin{bmatrix} q & p \\ q & p \end{bmatrix} + \frac{(1-p-q)^m}{p+q}\begin{bmatrix} p & -p \\ -q & q \end{bmatrix}$$

by showing that $\mathbf{PT} = \mathbf{T\Lambda}$, where $\mathbf{T}$ consists of eigenvectors of $\mathbf{P}$ and $\mathbf{\Lambda}$ is a diagonal matrix with the eigenvalues of $\mathbf{P}$ on the diagonal.

b) Let Y_n denote any Markov chain with stationary transition probabilities and transition probability matrix $\mathbf{P}$. Show that the m-step transition probability matrix $\mathbf{P}^{(m)} = \big(P_{ij}^{(m)}\big)$, where $P_{ij}^{(m)} = \text{Prob}\big(Y_{n+m} = j | Y_n = i\big)$, is determined by $\mathbf{P}^{(m)} = \mathbf{P}^m$, $m = 1, 2, \ldots$.

c) Discuss the properties of the ON/OFF system as depending on the values of p and q.

For which values of p and q does the Markov chain have limiting probabilities? If so, determine these probabilities.

d) Explain the concept of a stationary distribution. Write down the equation the stationary distribution must satisfy for the ON/OFF system, and solve it.

What is the connection between the limiting probabilities and a stationary distribution?

Problem 2 - Gas Production Platform

On a production platform for gas in the North Sea, the gas is compressed/condensed before it is transported to the mainland through pipelines. Assume that three identical compressors are installed on the platform for this purpose, but that at most two of them can be in operation at the same time. The following rules concerning repair and replacement applies:

- When a operating compressor fails, it is immediately repaired if another compressor is not already under repair. Note that this implies that at most one compressor can be under repair.
- Compressors that fail while another is repaired are put in a waiting queue.
- As soon as a compressor is repaired, it is put into production if two are not already in work. In this case the repaired compressor does not enter production until one of the operating compressors fail.
- When the repair of a compressor is finished, the repair work of one of the compressors in the waiting queue is immediately started if the waiting queue is not empty.

The compressors are monitored continuously from time $t = 0$. Let $X(t)$ denote the number of compressors that at time t are either under repair or in the waiting queue for repair. We shall now model $X(t)$, $t \geq 0$, as a birth and death process with the following birth and death rates: $\lambda_0 = \lambda_1 = 2\alpha$, $\lambda_2 = \alpha$, $\lambda_j = 0$ for $j \geq 3$, and $\mu_0 = 0$, $\mu_1 = \mu_2 = \mu_3 = \beta$, $\mu_j = 0$ for $j \geq 4$.

a) Explain briefly the main assumptions underlying the adopted model, which have not already been stated.

Also, explain how these assumptions lead to the given transition rates.

b) Let $P_{ij}(t) = \text{Prob}\big(X(t) = j \,|\, X(0) = i\big)$. Establish Kolmogorov's forward equations for $P_{ij}(t)$ for $i, j = 0, 1, 2, 3$ and $t \geq 0$.

c) Determine the stationary distribution $\pi = \big(\pi_0, \pi_1, \pi_2, \pi_3\big)$ of the process.

For a general birth and death process with birth rates λ_n and death rates μ_n, let T_i denote the time to reach state $i+1$ starting in state i. It has been shown in the textbook that $\text{E}[T_0] = \frac{1}{\lambda_0}$, and for $i \geq 1$,

$$\text{E}[T_i] = \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i}\,\text{E}[T_{i-1}]\,.$$

The compressor system is said to be functioning at time t if at least one compressor is working, that is, if $X(t) \leq 2$. Let T denote the point in time when the system ceases to function for the first time after $t = 0$, that is, $T = \min\{t \geq 0 | X(t) = 3\}$.

d) Establish and justify a system of equations to determine $m_i = \text{E}[T \,|\, X(0) = i]$ for $i = 0, 1, 2$. Calculate m_0, m_1, m_2 from these equations.

e) The compressor system has at the time we start our monitoring ($t = 0$) already been in work for a long time so that stationary conditions can be assumed. Find in this case the expected value of T when it is given that the system is functioning at (our) $t = 0$. Give your answer in terms of the π_i and m_i. You are not requested to substitute values for these.

Problem 3 - A Queueing System

Consider a birth and death process with birth rates $\lambda_n = \alpha\, q^n$, $n = 0, 1, 2, \ldots$, and death rates $\mu_0 = 0$, $\mu_n = \mu$, $n = 1, 2, \ldots$, where $\alpha > 0$, $\mu > 0$ and $0 < q < 1$.

a) Determine the stationary distribution of this process. Do you need to impose restrictions on the ratio α/μ for this stationary distribution to exist?

Explain briefly how this process can be interpreted as a queueing process. In particular, explain the effect of the birth rate λ_n.

Formulas for TMA4265 Stochastic Processes :

The law of total probability

Let $B_1, B_2, \ldots$ be pairwise disjoint events with $P(\cup_{i=1}^{\infty} B_i) = 1$. Then

$$P(A|C) = \sum_{i=1}^{\infty} P(A|B_i \cap C)P(B_i|C),$$
$$E[X|C] = \sum_{i=1}^{\infty} E[X|B_i \cap C]P(B_i|C).$$

Discrete time Markov chains

Chapman-Kolmogorov equations

$$P_{ij}^{(m+n)} = \sum_{k=0}^{\infty} P_{ik}^{(m)} P_{kj}^{(n)}.$$

For an irreducible and ergodic Markov chain, $\pi_j = \lim_{n\to\infty} P_{ij}^n$ exist and is given by the equations

$$\pi_j = \sum_i \pi_i P_{ij} \quad \text{og} \quad \sum_i \pi_i = 1.$$

For transient states i, j and k, the expected time spent in state j given start in state i, s_{ij}, is

$$s_{ij} = \delta_{ij} + \sum_k P_{ik} s_{kj}.$$

For transient states i and j, the probability of ever returning to state j given start in state i, f_{ij}, is

$$f_{ij} = (s_{ij} - \delta_{ij})/s_{jj}.$$

The Poisson process

The waiting time to the n-th event (the n-th arrival time), S_n, has the probability density

$$f_{S_n}(t) = \frac{\lambda^n t^{n-1}}{(n-1)!} e^{-\lambda t} \quad \text{for} \quad t \geq 0.$$

Given that the number of events $N(t) = n$, the arrival times $S_1, S_2, \ldots, S_n$ have the joint probability density

$$f_{S_1,S_2,\ldots,S_n|N(t)}(s_1, s_2, \ldots, s_n|n) = \frac{n!}{t^n} \quad \text{for} \quad 0 < s_1 < s_2 < \ldots < s_n \leq t.$$

Markov processes in continuous time

A (homogeneous) Markov process $X(t)$, $0 \le t \le \infty$, with state space $\Omega \subseteq \mathbf{Z}^+ = \{0, 1, 2, \ldots\}$, is called a birth and death process if

$$
\begin{aligned}
P_{i,i+1}(h) &= \lambda_i h + o(h) \\
P_{i,i-1}(h) &= \mu_i h + o(h) \\
P_{i,i}(h) &= 1 - (\lambda_i + \mu_i)h + o(h) \\
P_{ij}(h) &= o(h) \quad \text{for } |j - i| \ge 2
\end{aligned}
$$

where $P_{ij}(s) = P(X(t+s) = j | X(t) = i)$, $i, j \in \mathbf{Z}^+$, $\lambda_i \ge 0$ are birth rates, $\mu_i \ge 0$ are death rates.

The Chapman-Kolmogorov equations

$$P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t) P_{kj}(s).$$

Limit relations

$$\lim_{h \to 0} \frac{1 - P_{ii}(h)}{h} = v_i \,, \quad \lim_{h \to 0} \frac{P_{ij}(h)}{h} = q_{ij} \,, \ \ i \ne j$$

Kolmogorov's forward equations

$$P'_{ij}(t) = \sum_{k \ne j} q_{kj} P_{ik}(t) - v_j P_{ij}(t).$$

Kolmogorov's backward equations

$$P'_{ij}(t) = \sum_{k \ne i} q_{ik} P_{kj}(t) - v_i P_{ij}(t).$$

If $P_j = \lim_{t \to \infty} P_{ij}(t)$ exist, P_j are given by

$$v_j P_j = \sum_{k \ne j} q_{kj} P_k \quad \text{og} \quad \sum_j P_j = 1.$$

In particular, for birth and death processes

$$P_0 = \frac{1}{\sum_{k=0}^{\infty} \theta_k} \quad \text{og} \quad P_k = \theta_k P_0 \quad \text{for} \quad k = 1, 2, \ldots$$

where

$$\theta_0 = 1 \quad \text{og} \quad \theta_k = \frac{\lambda_0 \lambda_1 \cdot \ldots \cdot \lambda_{k-1}}{\mu_1 \mu_2 \cdot \ldots \cdot \mu_k} \quad \text{for} \quad k = 1, 2, \ldots$$

Queueing theory

For the average number of customers in the system L, in the queue L_Q; the average amount of time a customer spends in the system W, in the queue W_Q; the service time S; the average remaining time (or work) in the system V, and the arrival rate λ_a, the following relations obtain

$$L = \lambda_a W.$$

$$L_Q = \lambda_a W_Q.$$

$$V = \lambda_a E[SW_Q^*] + \lambda_a E[S^2]/2.$$

Some mathematical series

$$\sum_{k=0}^{n} a^k = \frac{1 - a^{n+1}}{1 - a} \quad , \qquad \sum_{k=0}^{\infty} k a^k = \frac{a}{(1-a)^2} \quad ,$$

Differential equation

The differential equation $f'(t) + \alpha f(t) = g(t)$ for $t \geq 0$ with initial condition $f(0) = a$ has the solution

$$f(t) = a e^{-\alpha t} + \int_0^t e^{-\alpha(t-s)} g(s)\, ds$$