Norges teknisk–
naturvitenskapelige universitet
Institutt for matematiske fag

English

Contact during exam:
Arvid Næss 73 59 70 53/ 99 53 83 50

EXAM IN COURSE TMA4265 STOCHASTIC PROCESSES

Monday, December 13, 2010
Time: 9:00–13:00

Permitted aids:

- Yellow A-5 sheet with your own handwritten notes (stamped by the Department of Mathematical Sciences)
- *Tabeller og formler i statistikk*, Tapir Forlag
- K. Rottmann: *Matematisk formelsamling*
- Calculator HP30S

The examination results are due January 13, 2011

Problem 1

Let $\{Y_n\}_{n=1}^{\infty}$ be a sequence of independent and identically distributed random variables which assume values among the non-negative integers:

$$P(Y_n = i) = p_i\,,\ \ i = 0, 1, 2, \dots\,,$$

where $0 < p_i < 1$ and $\sum_{i=0}^{\infty} p_i = 1$.

Define the sequence of random variables $\{X_n\}_{n=1}^{\infty}$ by the equation,

$$X_n = \min\{Y_1, Y_2, \dots, Y_n\}$$

.

a) Show that $\{X_n\}_{n=1}^{\infty}$ is a Markov chain, and determine its transition probability matrix.

b) Explain briefly the following two concepts related to a Markov chain: Recurrent state and transient state.

Which of the states of X_n are recurrent and which are transient? Does the Markov chain contain any ergodic states?

Find out if X_n has a limiting distribution. If so, write it down.

Problem 2

In this problem we shall consider a computer program which is assumed to contain n errors or bugs. (Of course, n is usually unknown.) Before the program is released for use, it is subjected to extensive and thorough testing. During this process, it is assumed that when an error is detected it is corrected immediately and without introducing new errors.

Let $t = 0$ correspond to the time when the testing is started, and let $X(t)$ denote the remaining number of errors in the program at time t for $t \geq 0$. Hence, $X(0) = n$.

We shall now introduce the following assumptions, which correspond to the assumptions made in the so-called Jelinski-Moranda model for software reliability.

- An error not detected at time t will be detected and corrected during the time interval $(t, t+h]$ with probability $\theta h + o(h)$.
- The errors are detected and corrected independently of each other.

a) Explain briefly why the process $X(t)$ is a Markov process.

Specify the state space of the process and show in detail that the transition rates q_{ij} are given as follows:

$$q_{i,i-1} = i\theta\,, \quad \text{for} \quad i = 1, 2, \dots, n$$

$$q_{ij} = 0\,, \quad \text{for} \quad i = 0, 1, \dots, n\,; \; j \neq i-1$$

The process $X(t)$ is thereby a pure death process.

b) Find the expected time from $t = 0$ until the first error is detected. Then, find the expected time until all errors have been detected.

With our usual notation, let $P_{ni}(t) = P\big(X(t) = i \,|\, X(0) = n\big)$. These transition probabilities can be determined by setting up and solving Kolmogorov's forward equations. However, the task now is to show that these probabilities can be determined in a different way.

Imagine that the n errors in the program at time $t = 0$ are numbered as $k = 1, 2, \ldots, n$. Then, let

$$X_k(t) = 1\,, \text{ if error no. } k \text{ is present at time } t\,,$$

while

$$X_k(t) = 0\,, \text{ if error no. } k \text{ has been detected before time } t\,.$$

c) Explain why $X_k(t)$ for $t \geq 0$ is a Markov process with state space $\{0, 1\}$.

Set up the transition rates for $X_k(t)$.

Find $P(X_k(t) = 1)$ for $t \geq 0$.

d) Express $X(t)$ in terms of the $X_k(t)$, $k = 1, \ldots, n$, and explain why $X(t)$ for each t ($t \geq 0$) is binomially distributed. Show that

$$P_{ni}(t) = \binom{n}{i} e^{-i\theta t}\big(1 - e^{-\theta t}\big)^{n-i}$$

for $i = 0, 1, \ldots, n$.

e) Assume now that the number of errors $X(0)$ in the computer program at the start of the testing, is Poisson distributed with parameter λ.

Determine the probability that all errors have been detected before time t.

Problem 3

a) Explain briefly the concept of a queueing system. In particular, what do we understand by an M/M/k-system and by an M/G/k-system ($1 \leq k \leq \infty$)?

Henceforth we are going to study a simple queueing model for a self-service station. Let $X(t)$ denote the number of customers in this queueing system at time t. We shall assume that potential customers arrive at the service station according to a Poisson process with intensity parameter $\lambda\,(> 0)$. However, it will be assumed that the desire to join the queue decreases as the queue increases. To model this we put

$$P(\text{Person arriving at time } t \text{ joins the queue} \,|\, X(t) = k) = \frac{1}{k+1}$$

It is assumed that the service time is exponentially distributed with parameter $\mu\,(> 0)$, and is independent of the arrivals. A person who decides not to join the queue is considered to be lost for the queueing system.

b) Let

$$P_{ij}(h) = P(X(t+h) = j | X(t) = i)\,.$$

Write down $P_{ij}(h)$ for small values of h. Specify in particular the birth and death parameters for the process $X(t)$.

c) Determine the stationary distribution for $X(t)$. Do you recognize it? What is the proportion of time that the service unit is vacant (empty) in the long run?

Formulas for TMA4265 Stochastic Processes :

The law of total probability

Let $B_1, B_2, \ldots$ be pairwise disjoint events with $P(\cup_{i=1}^{\infty} B_i) = 1$. Then

$$P(A|C) = \sum_{i=1}^{\infty} P(A|B_i \cap C)P(B_i|C),$$
$$E[X|C] = \sum_{i=1}^{\infty} E[X|B_i \cap C]P(B_i|C).$$

Discrete time Markov chains

Chapman-Kolmogorov equations

$$P_{ij}^{(m+n)} = \sum_{k=0}^{\infty} P_{ik}^{(m)} P_{kj}^{(n)}.$$

For an irreducible and ergodic Markov chain, $\pi_j = \lim_{n\to\infty} P_{ij}^n$ exist and is given by the equations

$$\pi_j = \sum_i \pi_i P_{ij} \quad \text{og} \quad \sum_i \pi_i = 1.$$

For transient states i, j and k, the expected time spent in state j given start in state i, s_{ij}, is

$$s_{ij} = \delta_{ij} + \sum_k P_{ik} s_{kj}.$$

For transient states i and j, the probability of ever returning to state j given start in state i, f_{ij}, is

$$f_{ij} = (s_{ij} - \delta_{ij})/s_{jj}.$$

The Poisson process

The waiting time to the n-th event (the n-th arrival time), S_n, has the probability density

$$f_{S_n}(t) = \frac{\lambda^n t^{n-1}}{(n-1)!} e^{-\lambda t} \quad \text{for} \quad t \geq 0.$$

Given that the number of events $N(t) = n$, the arrival times $S_1, S_2, \ldots, S_n$ have the joint probability density

$$f_{S_1,S_2,\ldots,S_n|N(t)}(s_1, s_2, \ldots, s_n|n) = \frac{n!}{t^n} \quad \text{for} \quad 0 < s_1 < s_2 < \ldots < s_n \leq t.$$

Markov processes in continuous time

A (homogeneous) Markov process $X(t)$, $0 \le t \le \infty$, with state space $\Omega \subseteq \mathbf{Z}^+ = \{0, 1, 2, \ldots\}$, is called a birth and death process if

$$P_{i,i+1}(h) = \lambda_i h + o(h)$$
$$P_{i,i-1}(h) = \mu_i h + o(h)$$
$$P_{i,i}(h) = 1 - (\lambda_i + \mu_i)h + o(h)$$
$$P_{ij}(h) = o(h) \quad \text{for } |j - i| \ge 2$$

where $P_{ij}(s) = P(X(t+s) = j | X(t) = i)$, $i, j \in \mathbf{Z}^+$, $\lambda_i \ge 0$ are birth rates, $\mu_i \ge 0$ are death rates.

The Chapman-Kolmogorov equations

$$P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t) P_{kj}(s).$$

Limit relations

$$\lim_{h \to 0} \frac{1 - P_{ii}(h)}{h} = v_i \,, \quad \lim_{h \to 0} \frac{P_{ij}(h)}{h} = q_{ij} \,, \;\; i \ne j$$

Kolmogorov's forward equations

$$P'_{ij}(t) = \sum_{k \ne j} q_{kj} P_{ik}(t) - v_j P_{ij}(t).$$

Kolmogorov's backward equations

$$P'_{ij}(t) = \sum_{k \ne i} q_{ik} P_{kj}(t) - v_i P_{ij}(t).$$

If $P_j = \lim_{t \to \infty} P_{ij}(t)$ exist, P_j are given by

$$v_j P_j = \sum_{k \ne j} q_{kj} P_k \quad \text{og} \quad \sum_j P_j = 1.$$

In particular, for birth and death processes

$$P_0 = \frac{1}{\sum_{k=0}^{\infty} \theta_k} \quad \text{og} \quad P_k = \theta_k P_0 \quad \text{for} \quad k = 1, 2, \ldots$$

where

$$\theta_0 = 1 \quad \text{og} \quad \theta_k = \frac{\lambda_0 \lambda_1 \cdot \ldots \cdot \lambda_{k-1}}{\mu_1 \mu_2 \cdot \ldots \cdot \mu_k} \quad \text{for} \quad k = 1, 2, \ldots$$

Queueing theory

For the average number of customers in the system L, in the queue L_Q; the average amount of time a customer spends in the system W, in the queue W_Q; the service time S; the average remaining time (or work) in the system V, and the arrival rate λ_a, the following relations obtain

$$L = \lambda_a W.$$

$$L_Q = \lambda_a W_Q.$$

$$V = \lambda_a E[SW_Q^*] + \lambda_a E[S^2]/2.$$

Some mathematical series

$$\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a} \quad , \qquad \sum_{k=0}^{\infty} k a^k = \frac{a}{(1-a)^2} \quad ,$$

Differential equation

The differential equation $f'(t) + \alpha f(t) = g(t)$ for $t \geq 0$ with initial condition $f(0) = a$ has the solution

$$f(t) = ae^{-\alpha t} + \int_0^t e^{-\alpha(t-s)} g(s)\, ds$$