TMA4265 Stochastic processes

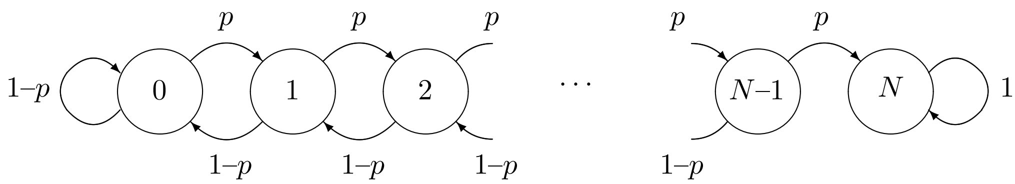
Semester project fall 2010

Problem 1

We shall have a look at a particular version of the gambler's ruin problem. Let X_n, $n \geq 0$ be a discrete-time Markov chain with state space $\Omega = \{0, 1, 2, \ldots, N\}$ and with transition probabilities

$$\mathrm{P}_{ij} = \mathrm{P}\{X_{n+1} = j | X_n = i\} = \begin{cases} p & \text{if } j = i+1 \text{ and } i = 0, 1, 2, \ldots, N-1, \\ 1-p & \text{if } j = i-1 \text{ and } i = 1, 2, \ldots, N-1, \\ 1-p & \text{if } i = j = 0, \\ 1 & \text{if } i = j = N, \\ 0 & \text{otherwise.} \end{cases}$$

We shall assume throughout that $0 < p < 1$. In contrast to the gambler's ruin problem that you know from the textbook, you may notice that state 0 is not absorbing. The Markov chain can also be illustrated by the following communication diagramme.



In the subproblems below you are supposed to introduce the necessary notation and concepts to express mathematically what is asked for (wherever it makes sense). Then you are asked to find the answer to the question posed.

a) Determine the Markov chain's equivalence classes. For each state, determine its period, and whether the state is recurrent or transient. Does $\lim_{n\to\infty} \mathrm{P}(X_n = j | X_0 = i)$ exist? If yes, what is the limit? All answers must be justified!

b) What is the expected time until the Markov chain reaches state N the first time, given that the Markov chain starts in state i at time 0?

c) What is the expected number of times that the Markov chain visits state 0, given that the Markov chain starts in state i at time 0?

d) Write MATLAB code (or R code, if you prefer) to simulate the Markov chain described above. (The command 'rand(1)' in MATLAB returns a random number which is uniformly distributed on the interval $[0, 1]$.) For $p = 0.5$, $X_0 = 20$ and $N = 200$ simulate the Markov chain several times (for example 500 times). For each of the simulations, extract the time when the chain arrives at state N the first time, how many times the chain visits state 0, and how many times the chain visits state 100. Use the simulted values to make 95% confidence intervals for the quantities asked for in point **b**) and **c**), and compare with the analytical answers you found there. Comment on the results. (Remember to include in your report the MATLAB code you have used for the simulations. To check on your calculations in **b**) and **c**), it might be useful to run simulations also for other values of p.) Finally, show a couple of examples of simulated time histories.

e) Use the simulation results in **d**) to also estimate (i) the probability that the time for first visit to state N is greater than or equal to $10,000$, and (ii) the probability that state 0 is visited fewer times than state 100. [Use still $p = 0.5$, $X_0 = 20$ og $N = 200$.]

Problem 2

This problem is a modified version of Example 4.38 in the textbook.

A component of a system is replaced regularly in the following way: When the component fails, it is replaced with a new one at the beginning of the next work day. If the component has been in use for $N > 1$ days, it is replaced even if it still works. Let $X_n = i$ if the component in use at the start of day n is in its i'th day of use, that is, if its present 'age' is i. For example, if a component fails on day $n - 1$, then a new component will be bin use on day n, that is, $X_n = 1$. We assume that each component, independently of each other, has life time equal to i with probability p_i, $i \geq 1$. If we let L denote the life time of an arbitrary component, then $P(L = i) = p_i$.

a) Show that X_n, $n \geq 1$, is a Markov chain with transition probabilities determined by (i $= 1, \ldots, N - 1$)

$$P_{i,1} = P(L = i | L \geq i) = \frac{P(L = i)}{P(L \geq i)}, \tag{1}$$

and

$$P_{i,i+1} = 1 - P_{i,1}. \tag{2}$$

Which value does $P_{N,1}$ have?

b) Classify the states of the Markov chain, and show that it has a stationary distribution π_i, $i = 1, \ldots, N$. (A short argument.) Derive the expression for this distribution.

c) Establish the expressions for the transition probabilities Q_{ij} for the reversed Markov chain, and show that $\pi_i \, P_{ij} = \pi_j \, Q_{ji}$.

Henceforth we shall assumet that $N = 50$ and

$$p_i = c \cdot 0.9^{|30-i|}, \quad i = 1, 2, \ldots, \infty$$

d) Determine the value of c, aand show that $P_{i,i+1} = 0.9$ for $30 \leq i \leq 49$. Write MATLAB code to calculate the transition probability matrix $\mathbb{P} = (P_{ij})$. Explain why you can use $\mathbb{P}^m$ for increasing m to find the stationary distribution. Do it, and verify by a suitable use of MATLAB that the results are in agreement with what you have found earlier (should be documented with MATLAB code).

e) Write MATLAB code to simulate the Markov chain. Show a couple of examples of simulated time histories, for example the first 500 days. Use time histories to check the values with π_i from point **d**). Do this for the two alternative interpretations of π_i. Show the results in graphical form.