TMA4265 Stochastic processes

Semester project fall 2011

Problem 1

We shall have a look at a generalized random walk with a finite state space. In particular, let X_n, $n \geq 0$ be a discrete-time Markov chain with state space $\Omega = \{0, 1, 2, \ldots, N\}$ and with transition probabilities

$$\mathrm{P}_{ij} = \mathrm{P}\{X_{n+1} = j | X_n = i\} = \begin{cases} r_0 & \text{if } j = i = 0 \\ p_0 & \text{if } j = 1 \text{ and } i = 0 \\ q & \text{if } j = i - 1 \text{ and } i = 1, 2, \ldots, N-1, \\ r & \text{if } j = i \text{ and } i = 1, 2, \ldots, N-1, \\ p & \text{if } j = i + 1 \text{ and } i = 1, 2, \ldots, N-1, \\ p_0 & \text{if } j = N-1 \text{ and } i = N \\ r_0 & \text{if } j = i = N \\ 0 & \text{otherwise.} \end{cases}$$

We shall assume throughout that $p > 0$, $q > 0$, $r \geq 0$, $r_0 \geq 0$.

In the subproblems below you are supposed to introduce the necessary notation and concepts to express mathematically what is asked for (wherever it makes sense). Then you are asked to find the answer to the question posed.

a) Determine the types of Markov chain that our model can give rise to. For each of these types: Determine the chain's equivalence classes. For each state, determine its period, and whether the state is recurrent or transient. Does $\lim_{n\to\infty} \mathrm{P}(X_n = j | X_0 = i)$ exist? If yes, what is the limit? All answers must be justified!

For the rest of Problem 1, $N = 100$, $p = q = 0.5$ and $r_0 = 0.5$ and 1.0.

b) For $r_0 = 0.5$, what is the expected time until the Markov chain reaches state 100 the first time, given that the Markov chain starts in state i at time 0 for $i = 25, 50, 75$?

c) What is the probability that the Markov chain visits state 100, given that the Markov chain starts in state i at time 0 for $i = 25, 50, 75$?

d) Write MATLAB code (or R code, if you prefer) to simulate the Markov chains described above. (The command 'rand(1)' in MATLAB returns a random number which is uniformly distributed on the interval $[0, 1]$.) For $r_0 = 0.5$ and 1.0, $X_0 = 25, 50, 75$, simulate the Markov chains many times (for example, at least 500 times). Use the simulated chains to estimate the quantities asked for in point **b**) and **c**). And use the simulated values to make 95% confidence intervals for the estimated quantities. Compare with the answers you found there. Comment on the results. Remember to include in your report the MATLAB code you have used for the simulations. Finally, show some examples of simulated time histories.

e) Use the simulation results in **d**) to also estimate the probability that the time for first visit to state 100 is greater than or equal to 1000. [Use still $X_0 = 25, 50, 75$.]

Problem 2

In this problem we shall study a discrete-time Markov chain X_n, $n \geq 0$, with state space $\Omega = \{0, 1, 2, 3, 4, 5\}$, and with transition probability matrix $\mathbf{P} = \big(P_{ij}\big)$, where

$$\mathbf{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{3} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

a) Determine the chain's equivalence classes. For each state, determine its period, and whether the state is recurrent or transient.

b) What can you say about $\lim_{n\to\infty} \mathbf{P}^{(n)}$.

c) Determine (by hand calculations) the long run proportion of time that the chain spends in the different states as depending on the initial state. That is, determine $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} P_{ij}^{(k)}$ for each pair i, j, where $\mathbf{P}^{(k)} = \big(P_{ij}^{(k)}\big)$. Explain your calculations.

d) Verify your results in **c**) in two ways. 1) By calculating the matrix $\mathbf{L}_n = \frac{1}{n} \sum_{k=1}^{n} \mathbf{P}^{(k)}$ for a suitably large n. 2) Partly by simulations for the initial condition $X_0 = 2$.