TMA4265 Stochastic processes

Semester project fall 2012

Problem 1

We shall have a look at a generalized random walk with an infinite state space. In particular, let X_n, $n \geq 0$ be a discrete-time Markov chain with state space $\Omega = \{0, 1, 2, \ldots\}$ and with transition probabilities

$$\mathrm{P}_{ij} = \mathrm{P}\{X_{n+1} = j | X_n = i\} = \begin{cases} 1-p & \text{if } j = i = 0 \\ 1-p^{i+1} & \text{if } j = i-1 \text{ and } i = 1, 2, \ldots, \\ p^{i+1} & \text{if } j = i+1 \text{ and } i = 0, 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

We shall assume throughout that $0 < p < 1$.

In the subproblems below you are supposed to introduce the necessary notation and concepts to express mathematically what is asked for (wherever relevant). Then you are asked to find the answer to the question posed.

a) Discuss the properties and characteristics of the Markov chain. You may also use the following result: An irreducible, aperiodic Markov chain is ergodic if the equilibrium equations has a non-negative normalizable solution (a probability distribution). You are also invited to use heuristic reasoning to argue why you believe that the states are positive recurrent. Use your results to argue why $\lim_{n\to\infty} \mathrm{P}(X_n = j | X_0 = i)$ exist, and provide explicit expressions for the limits.

For the rest of Problem 1, $p = 0.75$, and 0.9.

b) What is the expected time until the Markov chain reaches state 10 the first time, given that the Markov chain starts in state i at time 0 for $i = 0, \ldots, 9$? Discuss your results.

c) What is the probability that the Markov chain ever visits state 100, given that the Markov chain starts in state i at time 0 for any $i = 0, 1, 2, \ldots$? Warning: Don't try to calculate it, just use a nice, simple argument.

d) Write MATLAB code (or R code, if you prefer) to simulate the Markov chains described above. (The command 'rand(1)' in MATLAB returns a random number which is uniformly distributed on the interval $[0, 1]$.) Simulate the Markov chains so that you can provide answers to the estimation problems at hand. Use the simulated chains to verify your answer for the limiting probabilities in point **a**) for the first 11 states. For $p = 0.9$ estimate the quantities asked for in point **b**). Also estimate the probability that the time for first visit to state 15 is greater than or equal to 1000 for $i = 0$ and $i = 10$. If you can, use the simulated values to make 95% confidence intervals for the estimated quantities. Comment on the results. Remember to include in your report the MATLAB code you have used for the simulations. Finally, show some examples of simulated time histories.

Problem 2

Consider a branching process where X_n denotes the number of individuals in the n-th generation. We assume that the number of offspring that separate individulas have are mutually independent and follow the common probability distribution given by $p_j = \mathrm{P}($an individual has j offspring$)$, $j = 0, 1, \ldots$. Each individual lives for just one generation.

Given that $X_0 = 1$, you are now asked to investigate by numerical simulation how the branching process develops for a couple of distributions p_j. Remember to show examples of your simulations. We limit ourselves to cases where the expected number of offspring is less than or equal to 1. More specifically, we are interested in the following quantities: (i) The number of generations until the population dies out. (ii) The total number of individuals in the population ($\sum_n X_n$). (iii) The maximum number of individuals in a generation ($\max_n X_n$).

We start by looking at two different distributions for the number of offspring. The first is,

$$p_0 = 0.6 \quad , \quad p_1 = 0.05 \quad , \quad p_2 = 0.15 \quad \text{og} \quad p_3 = 0.2, \tag{1}$$

while the other is,

$$p_0 = 0.25 \quad , \quad p_1 = 0.60 \quad , \quad p_2 = 0.10 \quad \text{og} \quad p_3 = 0.05. \tag{2}$$

Note that in both cases the expected number of offspring for an individual is equal to 0.95.

a) Write Mathlab code for simulating the branching processes. For each of the two distributions p_j given above, estimate the mean value and standard deviation of X_n for $n = 10, 100, 1000$. How do your estimates compare with the analytical results? Try also to estimate the probability distributions for the quantities given in (i), (ii) og (iii). Compare the results for the two distributions p_j and comment briefly on what you see.

b) Now look at a situation where the expected number of offspring for an individual is equal to 1. Choose the distribution p_j yourself such that this is satisfied (with $p_j > 0, j = 0, \ldots, 3$). Simulate the resulting branching process and see if you can estimate the quantities asked for in **a**). Comment on the results and possible problems you encountered with your simulations.