TMA4265 Stochastic Processes

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Problem 1

a)

$X(t)$ becomes a birth and death process with birth rates $\lambda_i = (N-i)\lambda$ $(i = 0, 1, \ldots, N-1)$ and death rates $\mu_i = i\mu$ $(i = 1, 2, \ldots, N)$.

b)

For the limiting distribution to exist, $\lim_{t\to\infty} P'_{ij}(t) = 0$. This provides the following equations for the limiting distribution

$$0 = -\lambda N \pi_0 + \mu\pi_1$$

$$0 = (N-j+1)\lambda\pi_{j-1} - [(N-j)\lambda + j\mu]\pi_j + (j+1)\mu\pi j + 1\,,\ \ 1 \le j \le N-1$$

$$0 = \lambda\pi_{N-1} - N\mu\pi_N$$

By solving these equations, π_j may be expressed in terms of π_0. We find that $\pi_j = \binom{N}{j}\left(\frac{\lambda}{\mu}\right)^j \pi_0$.

By requiring that $\sum_{j=0}^{N} \pi_j = 1$, and using the result $\sum_{j=0}^{N}\binom{N}{j}x^j = (1+x)^N$, it is obtained that $\pi_0 = \big(\mu/(\lambda+\mu)\big)^N$. Finally, this yields the result

$$\pi_j = \binom{N}{j}\Big(\frac{\lambda}{\lambda+\mu}\Big)^j\Big(\frac{\mu}{\lambda+\mu}\Big)^{N-j}$$

c)

We may write $M(t+h) = \mathrm{E}[X(t+h)] = \mathrm{E}\big[\mathrm{E}[X(t+h)\,|\,X(t)]\big]$.
For small values of h, and neglecting $o(h)$ terms, it is seen that

$$\mathrm{E}[X(t+h)\,|\,X(t)=j] = (j-1)P_{j,j-1}(h) + jP_{jj}(h) + (j+1)P_{j,j+1}(h)$$

$$= (j-1)j\mu h + j\big(1 - ((N-j)\lambda + j\mu)h\big) + (j+1)(N-j)\lambda h$$
$$= j + N\lambda h - (\lambda+\mu)jh$$

This gives the equation

$$\mathrm{E}[X(t+h)\,|\,X(t)] = X(t) + N\lambda h - (\lambda+\mu)X(t)h$$

Taking expectations, and letting $h \to 0$, then leads to the desired equation:

$$M'(t) = N\lambda - (\lambda+\mu)M(t)$$

To solve this equation, we write

$$M'(t) + (\lambda+\mu)M(t) = N\lambda$$

It is seen that $e^{(\lambda+\mu)t}$ is an integrating factor: To solve this equation, we write

$$\Big(M(t)e^{(\lambda+\mu)t}\Big)' = N\lambda\, e^{(\lambda+\mu)t}$$

Integrating, using that $M(0) = i$, gives

$$M(t)e^{(\lambda+\mu)t} - i = \frac{N\lambda}{\lambda+\mu}\Big(e^{(\lambda+\mu)t} - 1\Big)$$

Hence,

$$M(t) = \frac{N\lambda}{\lambda+\mu} + \Big(i - \frac{N\lambda}{\lambda+\mu}\Big)e^{-(\lambda+\mu)t}$$

From this result we deduce the limit

$$\lim_{t\to\infty} M(t) = \frac{N\lambda}{\lambda+\mu}$$

Problem 2

a)

$Y(t) = N(t-a,t]$ $(t \geq a)$ because passengers that arrive in the time interval $(t-a,t]$ according to the assumptions, have not finished their service at time t. Those passengers that arrive before time $t-a$ have finished before time t and therefore do not contribute to $Y(t)$.

The random variable $N(t-a,t]$ is by the definition of a Poisson process, Poisson distributed with expectation λa. Hence,

$$Y(t) \sim Poisson(\lambda a)$$

.

For $0 < t < a$, the number of passengers at the terminals will be $N(0,t]$, and

$$Y(t) \sim Poisson(\lambda t)$$

since none of the passengers have finished their service at time t.

b)

We need to determine $P(Y(s) = 0 \cap Y(t) = 0)$
If $t - s < a$, there can be no arrivals in $(s - a, t]$. That is,

$$P(Y(s) = 0 \cap Y(t) = 0) = P(N(s - a, t] = 0) = e^{-\lambda(t-s+a)}$$

If $t - s \geq a$, there can be no arrivals in $(s - a, s]$ as well as no arrivals in $(t - a, t]$, and the two intervals are disjoint. That is,

$$P(Y(s) = 0 \cap Y(t) = 0) = P(N(s - a, s] = 0 \cap N(t - a, t] = 0) = e^{-\lambda a}\, e^{-\lambda a} = e^{-2\lambda a}$$

c)
Little's formula is given as $L = \lambda W$, where L is the average number of passengers in the system, and W is the average time a passenger spends at the terminal. For the present problem, it is clear that $W = m_A$. Hence, $L = \lambda m_A$. This corresponds to the result in point a), since $m_A = a$ and $\mathrm{E}[Y(t)] = \lambda a$ (for large t).

d)

Using the hint, we write

$$P(Y(t) = k) = \sum_{n=k}^{\infty} P(Y(t) = k \,|\, N(t) = n)\, P(N(t) = n)$$

where

$$P(Y(t) = k \,|\, N(t) = n) = \binom{n}{k} p^k (1 - p)^{n-k}$$

Here, p is the probability that an arbitrary arrival in the time interval $(0, t]$ still is at the terminal at time t.

$$p = \int_0^t P(\text{passenger is at terminal at time}\, t \,|\, \text{arrival at time}\, u)\, f_U(u)\, du$$

where $f_U(u)$ is the probability density of an arbitrary arrival time. This is uniform and it is given as $f_U(u) = 1/t$ for $0 \leq u \leq t$. This leads to the result

$$p = \int_0^t (1 - G(t-u)) \frac{1}{t} \, du = \frac{1}{t} \int_0^t (1 - G(s)) \, ds$$

Going back to the first equation, this may now be written in the following form

$$\begin{aligned}
P(Y(t) = k) &= \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} \frac{(\lambda t)^n}{n!} e^{-\lambda t} \\
&= \sum_{n=k}^{\infty} \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \frac{(\lambda t)^n}{n!} e^{-\lambda t} \\
&= \frac{(\lambda t p)^k}{k!} e^{-\lambda t} \sum_{n=k}^{\infty} \frac{(\lambda t (1-p))^{n-k}}{(n-k)!} \\
&= \frac{(\lambda t p)^k}{k!} e^{-\lambda t p}
\end{aligned}$$

This shows that $Y(t) \sim Poisson(\lambda t p) = Poisson(\lambda \int_0^t (1 - G(s)) \, ds)$

Problem 3

a)

The definition of a standard Brownian motion $B(t)$ is as follows:

1. $B(0) = 0$
2. $B(t)$ has stationary and independent increments
3. $B(t) \sim N(0, t)$ $(t > 0)$

b)

From the definition, we know that $B(2) \sim N(0, 2)$. Hence, it is obtained that

$$P(B(2) \geq 3) = 1 - \Phi(3/\sqrt{2}) = 1 - \Phi(2.12) = 0.017$$

Also, we have that

$$P(B(2) \geq 3 \,|\, B(1) = 1) = P(B(2) - B(1) \geq 2) = 1 - \Phi(2) = 0.023.$$

Here, the first equation is due to the property of independent increments, the second is due to stationary increments, that is, $B(2) - B(1) \sim B(1)$.

c)

We need to determine $P(T_3 \leq 2)$. Based on the hint, we apply the rule of total probability:

$$\begin{aligned} P(B(2) \geq 3) &= P(B(2) \geq 3 \,|\, T_3 \leq 2)\, P(T_3 \leq 2) + P(B(2) \geq 3 \,|\, T_3 > 2)\, P(T_3 > 2) \\ &= \frac{1}{2}\, P(T_3 \leq 2) \end{aligned}$$

Here, $P(B(2) \geq 3 \,|\, T_3 \leq 2) = 1/2$ because when given that the process has visited state 3 in the time interval $[0, 2]$, then the probability that the process is above 3 at time 2 must be equal to the probability of being below. Clearly, $P(B(2) \geq 3 \,|\, T_3 > 2) = 0$ Hence, it is obtained that

$$P(T_3 \leq 2) = 2\, P(B(2) \geq 3) = 2 \cdot 0.017 = 0.034$$