TMA4265 Stochastic Processes

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Problem 1

a)

It is seen that

$$X_{n+1} = \min\{X_n, Y_{n+1}\}$$

Hence, when X_n is given, the statistics of X_{n+1} does not depend on X_k for $k < n$ which implies that $\{X_n\}_{n=1}^{\infty}$ is a Markov chain. The transition probability matrix $\mathbf{P} = (P_{ij})$, where $P_{ij} = P(X_{n+1} = j \mid X_n = i)$ for $i, j = 0, 1, 2, \ldots$, (does not depend on n) is given as follows:

$$P_{00} = 1\,;\;\; P_{0j} = 0\,,\;\; j > 1$$

while for $i \geq 1$,

$$P_{ij} = \begin{cases} p_j\,,\;\; 0 < j < i \\ 1 - \sum_{j=0}^{i-1} p_j\,,\;\; j = i \\ 0\,,\quad j > i \end{cases}$$

b)

Let $f_i =$ The probability of ever returning to state i. i is a recurrent state if $f_i = 1$. i is a transient state if $f_i < 1$.

We now use the following results:

$$f_i = 1 \Leftrightarrow \sum_{m=1}^{\infty} P_{ii}^m = \infty\,,\;\; f_i < 1 \Leftrightarrow \sum_{m=1}^{\infty} P_{ii}^m < \infty\,,$$

where $P_{ij}^m = P(X_{n+m} = j \mid X_n = i)$.

Since $P_{00} = 1$, it follows that $P_{00}^m = 1$ for $m = 1, 2, \ldots$. Hence, $\sum_{m=1}^{\infty} P_{00}^m = \infty$, so $\{0\}$ is recurrent.

Also, since $P_{ij} = 0$ for $j > i$, it follows that $P_{ii}^m = (P_{ii})^m = (1 - \sum_{j=0}^{i-1} p_j)^m$ for $i \geq 1$ (since then the only way of returning to i is that the chain remains in i). Therefore,

$\sum_{m=1}^{\infty} P_{ii}^m = \sum_{m=1}^{\infty}(1-\sum_{j=0}^{i-1} p_j)^m < \infty$ since $(1-\sum_{j=0}^{i-1} p_j) < 1$, so the states $\{1, 2, 3, \ldots\}$ are transient.

Since $P_{00} = 1$, $\{0\}$ is an aperiodic state. Also, since $P_{00} = 1$, the average return time must be 1, so $\{0\}$ is positively recurrent. $\{0\}$ is therefore ergodic.

If the Markov chain $\{X_n\}_{n=1}^{\infty}$ has a limiting distribution $\boldsymbol{\pi} = (\pi_0, \pi_1, \ldots)$, it must satisfy the equations $\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}$, that is,

$$\pi_0 = \pi_0 + p_0 \sum_{i=1}^{\infty} \pi_i$$

and, for $i = 1, 2, \ldots$,

$$\pi_i = \pi_i \, (1 - \sum_{j=0}^{i-1} p_j) + p_i \sum_{j=i+1}^{\infty} \pi_j$$

The first equation leads to $p_0 \sum_{i=1}^{\infty} \pi_i = 0$, or equivalently, $\sum_{i=1}^{\infty} \pi_i = 0$. This immediately gives $\pi_1 = \pi_2 = \ldots = 0$. From the normalization condition $\sum_{i=0}^{\infty} \pi_i = 1$, it is obtained that $\pi_0 = 1$. It is now easy to verify that $\boldsymbol{\pi} = (1, 0, 0, \ldots)$ indeed satisfies $\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}$. It is therefore a limiting distribution.

Alternatively: Eventually, the chain will leave all transient states and end up in $\{0\}$ as the only recurrent state. This implies that the only possible limiting distribution is $\boldsymbol{\pi} = (1, 0, 0, \ldots)$, which is easily verified to satisfy $\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}$.

Problem 2

a)

If information about $X(t)$ for time $t \leq s$ is given, then the distribution of $X(t)$ for $t > s$ conditional on this information will depend only on the value of $X(s)$. That is, $X(t)$ is a Markov process. It has state space $\Omega = \{0, 1, \ldots, n\}$.

Obviously, for $j > i = 0, 1, \ldots, n-1$ and any $h > 0$:

$$P(X(t+h) = j | X(t) = i) = 0 \;\; \Rightarrow q_{ij} = 0$$

For $i = 1, 2, \ldots, n$ and $h > 0$:

$$\begin{aligned} P(X(t+h) = i-1 | X(t) = i) &= P\big(\text{Detecting one error during}(t, t+h)\big) \\ &= \binom{i}{1}(\theta h + o(h))(1 - \theta h - o(h))^{i-1} = i\theta h + o(h)\,, \end{aligned}$$

Hence,

$$q_{i,i-1} = \lim_{h\to 0} \frac{P(X(t+h) = i-1|X(t) = i)}{h} = i\theta \,,$$

For $j < i-1$, $i = 2, \ldots, n$ and $h > 0$:

$$\begin{aligned} P(X(t+h) = j|X(t) = i) &= P\big(\text{Detecting i - j errors during}(t, t+h)\big) \\ &= \binom{i}{i-j} (\theta h + o(h))^{i-j} (1 - \theta h - o(h))^j = o(h) \,, \end{aligned}$$

since $i - j \geq 2$. Hence,

$$q_{ij} = \lim_{h\to 0} \frac{P(X(t+h) = j|X(t) = i)}{h} = 0 \,,$$

And indeed, the process becomes a pure death process with parameters $\mu_i = i\,\theta$, for $i = 0, 1, \ldots, n$.

b)

The time $T_{(i)}$ until the first 'death' in state i is exponentially distributed with parameter $\mu_i = i\,\theta$. Hence, the expected time until the first error is detected is $\mathrm{E}[T_{(n)}] = 1/(n\theta)$.

The expected time until all errors have been detected is then

$$\mathrm{E}[T_{(1)} + T_{(2)} + \ldots + T_{(n)}] = \sum_{i=1}^{n} \frac{1}{i\,\theta} \,.$$

c)

The process $X_k(t)$ is just a special case of the process $X(t)$ discussed in a) with $n = 1$. Hence, $X_k(t)$ is a Markov process with state space $\{0, 1\}$.

The transition rates in this case are then $q_{10} = \theta$ and $q_{01} = 0$.

For a particular k, the time T_k until the error is detected is exponentially distributed with parameter θ. Hence,

$$P(X_k(t) = 1) = P(T_k > t) = e^{-\theta t} \,,\; t \geq 0$$

d)

Clearly,

$$X(t) = X_1(t) + X_2(t) + \ldots + X_n(t)$$

For each fixed t, the random variables $X_k(t)$, $k = 1, , \ldots, n$ becomes a set of independent, identically distributed Bernoulli variables with probability of 'success' $p = e^{-\theta t}$. Hence, $X(t)$ is binomially distributed, and

$$P_{ni}(t) = \binom{n}{i} p^i (1-p)^{n-i} = \binom{n}{i} e^{-i\theta t} \big(1 - e^{-\theta t}\big)^{n-i}$$

Note: $p = 1$ for $t = 0$.

e)

If the number of errors $X(0)$ is Poisson distributed with parameter λ, we can write

$$X(t) = X_1(t) + X_2(t) + \ldots + X_{X(0)}(t)\,,$$

Then, according to d),

$$\begin{aligned} P\big(X(t) = 0\big) &= \sum_{n=0}^{\infty} P\big(X(t) = 0 \,|\, X(0) = n\big)\, P\big(X(0) = n\big) = \\ &= \sum_{n=0}^{\infty} \big(1 - e^{-\theta t}\big)^n \frac{\lambda^n}{n!} e^{-\lambda} = e^{-\lambda} \sum_{n=0}^{\infty} \frac{\big(\lambda(1 - e^{-\theta t})\big)^n}{n!} \\ &= e^{-\lambda}\, e^{\lambda(1-e^{-\theta t})} = e^{-e^{-\theta t}}\,. \end{aligned}$$

Problem 3

a)

See the textbook.

b)

From the information given we have an M/M/1-system, and it follows that ($h > 0$),

$$P_{i,i+1}(h) = P(X(t+h) = i+1 | X(t) = i) = \frac{\lambda}{i+1} h + o(h)\,,\ \ i \geq 0$$

$$P_{i,i-1}(h) = P(X(t+h) = i-1 | X(t) = i) = \mu h + o(h)\,,\ \ i \geq 1$$

$$P_{ii}(h) = P(X(t+h) = i | X(t) = i) = 1 - \frac{\lambda}{i+1} h - \mu h + o(h)\,,\ \ i \geq 1$$

$$P_{00}(h) = P(X(t+h) = 0|X(t) = 0) = 1 - \lambda h + o(h)\,,$$

while

$$P_{ij}(h) = P(X(t+h) = j|X(t) = i) = o(h)\,,\ \ |j-i| > 1.$$

Hence, $X(t)$ is a birth and death process with birth rates $\lambda_i = \lambda/(i+1)$, for $i = 0, 1, \ldots$, and death rates $\mu_0 = 0$, $\mu_i = \mu$, $i = 1, 2, \ldots$.

c)

The stationary distribution is given as

$$P_i = \theta_i\, P_0\,,\ \ i = 1, 2, \ldots$$

where

$$P_0 = \frac{1}{1 + \sum_{i=1}^{\infty} \theta_i}\,,\ \ i = 1, 2, \ldots$$

and

$$\theta_i = \frac{\lambda_0 \cdots \lambda_{i-1}}{\mu_1 \cdots \mu_i} = \frac{1}{i!}\Big(\frac{\lambda}{\mu}\Big)^i,\ \ i = 1, 2, \ldots$$

This leads to the result:

$$P_i = \frac{1}{i!}\Big(\frac{\lambda}{\mu}\Big)^i e^{-\frac{\lambda}{\mu}}\,,\ \ i = 0, 1, 2, \ldots\,.$$

That is, the limiting distribution is a Poisson distribution with parameter λ/μ.

The proportion of time that the system is vacant in the long run is $P_0 = e^{-\frac{\lambda}{\mu}}$.