TMA4265 Stochastic Processes

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Solution sketch - Exam December 2011

Problem 1

a)

By construction, the probability of going from state i ($i = 0, 1$) to state j ($j = 0, 1$) is fully determined by the initial state i. Hence the process becomes a Markov chain. The transition probability matrix is given as,

$$\mathbf{P} = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}$$

The eigenvalues λ_1 and λ_2 are obtained by solving the equation $\det(\mathbf{P} - \lambda\mathbf{I}) = 0$, where det denotes the determinant and

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This leads to the equation $(1-p-\lambda)(1-q-\lambda) - pq = 0$, or $\lambda^2 - (2-p-q)\lambda + (1-p-q) = (\lambda-1)(\lambda-(1-p-q)) = 0$. The two eigenvalues are therefor $\lambda_1 = 1$ and $\lambda_2 = 1-p-q$. Corresponding eigenvectors are found from the equations, $\mathbf{P}\mathbf{v_1} = \mathbf{v_1}$ and $\mathbf{P}\mathbf{v_2} = (1-p-q)\mathbf{v_2}$. The first equation gives e.g. $\mathbf{v_1} = (1,1)^T$, while the second equation gives e.g. $\mathbf{v_2} = (p, -q)^T$. Define the matrix

$$\mathbf{T} = \begin{bmatrix} 1 & p \\ 1 & -q \end{bmatrix}$$

Then

$$\mathbf{PT} = \mathbf{T\Lambda} = \mathbf{T} \begin{bmatrix} 1 & 0 \\ 0 & 1-p-q \end{bmatrix},$$

which leads to,

$$\mathbf{P} = \mathbf{T\Lambda T}^{-1}$$

It then follows that

$$\mathbf{P}^m = \mathbf{T\Lambda}^m\mathbf{T}^{-1}, \; m = 1, 2, \ldots$$

This leads to the equation,

$$\mathbf{P}^m = \mathbf{T}\begin{bmatrix}1 & 0 \\ 0 & (1-p-q)^m\end{bmatrix}\mathbf{T}^{-1} = \mathbf{T}\begin{bmatrix}1 & 0 \\ 0 & 0\end{bmatrix}\mathbf{T}^{-1} + \mathbf{T}\begin{bmatrix}0 & 0 \\ 0 & (1-p-q)^m\end{bmatrix}\mathbf{T}^{-1}$$

It is found that,

$$\mathbf{T}^{-1} = \frac{1}{p+q}\begin{bmatrix}q & p \\ 1 & -1\end{bmatrix},$$

which, when combined with the previous equation, leads to,

$$\mathbf{P}^m = \frac{1}{p+q}\begin{bmatrix}q & p \\ q & p\end{bmatrix} + \frac{(1-p-q)^m}{p+q}\begin{bmatrix}p & -p \\ -q & q\end{bmatrix}$$

b)

This follows by using the Chapman-Kolmogorov equation (see the textbook).

c)

Clearly, for $0 < p, q < 1$ or $p = 1$ and $0 < q < 1$ or $q = 1$ and $0 < p < 1$, the Markov chain is irreducible and ergodic (aperiodic and positively recurrent) and therefore limiting probabilities exist. In this case $\lim_{m\to\infty}(1-p-q)^m \to 0$, which implies that

$$\lim_{m\to\infty}\mathbf{P}^m = \frac{1}{p+q}\begin{bmatrix}q & p \\ q & p\end{bmatrix}.$$

That is, the limiting probabilities are given as $\lim_{m\to\infty} P_{i0}^{(m)} = q/(p+q)$ for $i = 0, 1$ and $\lim_{m\to\infty} P_{i1}^{(m)} = p/(p+q)$ for $i = 0, 1$.

For $p = q = 1$ the Markov chain is still irreducible and positively recurrent, but periodic with period 2, and therefore limiting probabilities do not exist. This is easily verified from the expression for $\mathbf{P}^m$, since $\lim_{m\to\infty}(1-p-q)^m = \lim_{m\to\infty}(-1)^m$, which does not exist.

For $p = q = 0$ the Markov chain is reducible to two trivial irreducible subchains, which are both aperiodic and positively recurrent. The limiting probabilities obviously do not exist since e.g. $P_{00}^{(m)} = 1$ and $P_{10}^{(m)} = 0$ for any value of m.

d)

The distribution π is a stationary distribution for the Markov chain X_n if given that the distribution of X_0 is a stationary distribution π, then the distribution of X_n is also π for any $n = 1, 2, \ldots$.

The stationary distribution must satisfy the equation $\pi\,\mathbf{P} = \pi$, which for the present problem assumes the form

$$(\pi_1, \pi_2)\begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix} = (\pi_1, \pi_2)\,.$$

The solution must satisfy the equation $p\,\pi_1 = q\,\pi_2$ and $\pi_1 + \pi_2 = 1$. This gives the solution $\pi_1 = q/(p+q)$ and $\pi_2 = p/(p+q)$ provided $p + q > 0$. If $p = q = 0$, then any distribution (π_1, π_2) is a stationary distribution. Hence, a stationary distribution exists for any $0 \le p, q \le 1$.

It is seen that the limiting distribution when it exists, agrees with the stationary distribution, which is a general result. However, a Markov chain may have a stationary distribution even if the limiting distribution does not exist, as has been demonstrated here.

Problem 2

a)

In a birth and death process the time from each birth to the next is assumed to be exponentially distributed with the birth rate λ_n as parameter. Specifically, if the population size is $n \in \mathbf{Z}_+ = \{0, 1, 2, \ldots\}$, the birth rate parameter is λ_n, and the distribution of the time until the next birth happens is exponentially distributed with parameter λ_n. Similarly for the deaths: If the population size is n, and the death rate parameter is μ_n, then the distribution of the time until the next death happens is exponentially distributed with parameter μ_n. This implies that $\mu_0 = 0$.

According to the assumptions of the model in this problem, $\lambda_2 = \alpha$ implies that the operation time of each compressor until failure is exponentially distributed with parameter α. The birth rates λ_0 and λ_1 are then determined by the minimum of two operation times, which is exponentially distributed with parameter 2α. Hence, $\lambda_0 = \lambda_1 = 2\alpha$

b)

The general Kolmogorov's forward equations

$$P'_{ij}(t) = \sum_{k \neq j} q_{kj} P_{ik}(t) - v_j P_{ij}(t).$$

For the birth and death process here:

$$v_0 = 2\alpha\,,\; v_1 = 2\alpha + \beta\,,\; v_2 = \alpha + \beta\,,\; v_3 = \beta$$

and

$$P_{01} = 1\,,\; P_{10} = \frac{\beta}{2\alpha+\beta}\,,$$

$$P_{12} = \frac{2\alpha}{2\alpha+\beta}\,,\; P_{21} = \frac{\beta}{\alpha+\beta}\,,$$

$$P_{23} = \frac{\alpha}{\alpha+\beta}\,,\; P_{32} = 1\,,$$

This leads to,

$$q_{01} = q_{12} = 2\alpha\,,\; q_{10} = q_{21} = q_{32} = \beta\,,\; q_{23} = \alpha\,,$$

Kolmogorov's forward equations for the present model for $i = 0, 1, 2, 3$:

$$j = 0:\; P'_{i0}(t) = \beta P_{i1}(t) - 2\alpha P_{i0}(t)\,,$$

$$j = 1:\; P'_{i1}(t) = 2\alpha P_{i0}(t) + \beta P_{i2}(t) - (2\alpha+\beta)P_{i1}(t)\,,$$

$$j = 2:\; P'_{i2}(t) = 2\alpha P_{i1}(t) + \beta P_{i3}(t) - (\alpha+\beta)P_{i2}(t)\,,$$

$$j = 3:\; P'_{i3}(t) = \alpha P_{i2}(t) - \beta P_{i3}(t)\,,$$

c)

The stationary distribution can be obtained from these equations by putting $\lim_{t\to\infty} P'_{ij}(t) = 0$, and $\lim_{t\to\infty} P_{ij}(t) = \pi_j$. This leads to the equations:

$$j = 0:\; 0 = \beta\pi_1 - 2\alpha\pi_0\,,$$

$$j = 1:\; 0 = 2\alpha\pi_0 + \beta\pi_2 - (2\alpha+\beta)\pi_1\,,$$

$$j = 2:\; 0 = 2\alpha\pi_1 + \beta\pi_3 - (\alpha+\beta)\pi_2\,,$$

$$j = 3:\; 0 = \alpha\pi_2 - \beta\pi_3\,,$$

By solving these equations, it is obtained that $\pi_1 = (2\alpha/\beta)\,\pi_0$, $\pi_2 = (2\alpha/\beta)\,\pi_1 = (2\alpha/\beta)^2\,\pi_0$, $\pi_3 = (\alpha/\beta)\,\pi_2 = (\alpha/\beta)\,(2\alpha/\beta)^2\,\pi_0$. The condition $\sum_{j=0}^{3}\pi_j = 1$ then finally gives the solution:

$$\pi_0 = \frac{1}{\gamma}\,,\; \pi_1 = \frac{2\alpha}{\beta\gamma}\,,\; \pi_2 = \Big(\frac{2\alpha}{\beta}\Big)^2\frac{1}{\gamma}\,,\; \pi_3 = \Big(\frac{2\alpha}{\beta}\Big)^2\frac{\alpha}{\beta\gamma}\,,$$

where

$$\gamma = 1 + \frac{2\alpha}{\beta} + \Big(\frac{2\alpha}{\beta}\Big)^2\Big(1 + \frac{\alpha}{\beta}\Big)\,,$$

d)

Let $e_i = \mathrm{E}[T_i]$, then we get the equations $m_2 = e_2$, $m_1 = e_1 + e_2$ and $m_0 = e_0 + e_1 + e_2$, where

$$e_0 = \frac{1}{2\alpha}$$

$$e_1 = \frac{1}{2\alpha} + \frac{\beta}{2\alpha}\,e_0 = \frac{1}{2\alpha} + \frac{\beta}{(2\alpha)^2}$$

$$e_2 = \frac{1}{\alpha} + \frac{\beta}{\alpha}\,e_1 = \frac{1}{\alpha} + \frac{\beta}{\alpha}\Big(\frac{1}{2\alpha} + \frac{\beta}{(2\alpha)^2}\Big) = \frac{1}{\alpha} + \frac{1}{\alpha}\Big(\frac{\beta}{2\alpha} + \Big(\frac{\beta}{2\alpha}\Big)^2\Big)$$

This results in the following solutions:

$$m_0 = \frac{3}{2\alpha} + \Big(\frac{1}{\alpha} + \frac{1}{\beta}\Big)\Big(\frac{\beta}{2\alpha} + \Big(\frac{\beta}{2\alpha}\Big)^2\Big),$$

$$m_1 = \frac{1}{\alpha} + \Big(\frac{1}{\alpha} + \frac{1}{\beta}\Big)\Big(\frac{\beta}{2\alpha} + \Big(\frac{\beta}{2\alpha}\Big)^2\Big),$$

$$m_2 = \frac{1}{\alpha} + \frac{1}{\alpha}\Big(\frac{\beta}{2\alpha} + \Big(\frac{\beta}{2\alpha}\Big)^2\Big).$$

e)

You are asked to calculate $\mathrm{E}[T \,|\, X(0) \le 2]$ assuming that stationary conditions have been reached. Conditioning on the particular state at $t = 0$, we find,

$$\mathrm{E}[T \,|\, X(0) \le 2] = \sum_{i=0}^{3} \mathrm{E}[T \,|\, X(0) \le 2 \cap X(0) = i]\, P\big(X(0) = i \,|\, X(0) \le 2\big)$$

$$= \sum_{i=0}^{2} \mathrm{E}[T \,|\, X(0) = i]\, \frac{P\big(X(0) = i\big)}{P\big(X(0) \le 2\big)}.$$

$$= \sum_{i=0}^{2} m_i\, \frac{\pi_i}{\pi_0 + \pi_1 + \pi_2} = \frac{m_0\pi_0 + m_1\pi_1 + m_2\pi_2}{\pi_0 + \pi_1 + \pi_2}.$$

Problem 3

a)

From the given formulas, it follows that the stationary distribution P_j, $j = 0, 1, 2, \ldots$ is given as follows:

$$P_j = \theta_j\, P_0\,,\ \ j = 1, 2, \ldots$$

where

$$P_0 = \frac{1}{1 + \sum_{j=1}^{\infty} \theta_j},$$

and

$$\theta_j = \frac{\lambda_0 \cdots \lambda_{j-1}}{\mu_1 \cdots \mu_j} = \frac{\alpha^j \cdot q \cdot q^2 \cdot \ldots \cdot q^{j-1}}{\mu^j}$$
$$= \left(\frac{\alpha}{\mu}\right)^j q^{\sum_{i=1}^{j-1} i} = \left(\frac{\alpha}{\mu}\right)^j q^{\frac{j(j-1)}{2}}, \; j = 1, 2, \ldots$$

Hence,

$$P_j = \left(\frac{\alpha}{\mu}\right)^j q^{\frac{j(j-1)}{2}} P_0, \; j = 1, 2, \ldots$$

It is seen that,

$$\theta_j = \left(\frac{\alpha}{\mu}\right)^j q^{\frac{j(j-1)}{2}} = \left(\frac{\alpha}{\mu} q^{\frac{j-1}{2}}\right)^j, \; j = 1, 2, \ldots$$

For any given $\alpha > 0$ and $\mu > 0$ there is a j_0 such that

$$\frac{\alpha}{\mu} q^{\frac{j-1}{2}} < \frac{\alpha}{\mu} q^{\frac{j_0-1}{2}} < 1,$$

for $j > j_0$ since $0 < q < 1$. Hence, $\sum_{j=1}^{\infty} \theta_j < \infty$ for any choices of $\alpha > 0$ and $\mu > 0$, which is the condition for the existence of the stationary distribution.

From the discussion in point a) of Problem 2, it follows that the birth and death process considered here can be interpreted as an M/M/1 queueing system with exponential arrival rates $\lambda_n = \alpha\, q^n$ $(n \geq 0)$ and exponential departure rates $\mu_n = \mu$ $(n \geq 1)$.

The particular expression for the arrival rate implies that the more customers in the system the less likely it is that an arriving customer will enter the queueing system.