TMA4265 Stochastic Processes

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Solution sketch - Exam December 2012

Problem 1

a)

Let $P_{ij}(h) = P\big(X(t+h) = j \,|\, X(t) = i\big)$. Then

$$P_{13}(h) = P\big(\text{repair finished in } (t, t+h]\big) = \mu h + o(h).$$

$$P_{i3}(h) = o(h), \;\; i = 0, 2, 3.$$

$$P_{02}(h) = P_{13}(h) = \mu h + o(h).$$

$$P_{32}(h) = P\big(\text{one of three components fail during } (t, t+h]\big)$$

$$= \binom{3}{1}(\lambda h + o(h))(1 - \lambda h + o(h))^2 = 3\lambda h + o(h).$$

$$P_{i2}(h) = o(h), \;\; i = 1, 2.$$

$$P_{21}(h) = P\big(\text{replace component during } (t, t+h]\big) = \gamma h + o(h).$$

This follows since the time interval until replacement is finished is exponentially distributed with parameter γ.

$$P_{i1}(h) = o(h), \;\; i = 0, 1, 3.$$

$$P_{10}(h) = P_{32}(h) = 3\lambda h + o(h).$$

$$P_{i0}(h) = o(h), \;\; i = 0, 2, 3.$$

The transition rates:

$$q_{ij} = \lim_{h \to 0} \frac{P_{ij}(h)}{h}$$

From the relations above it then follows that,

$$q_{13} = q_{02} = \mu$$

$$q_{32} = q_{10} = 3\lambda$$

$$q_{21} = \gamma$$

This gives the communication diagramme shown in the figure.

b)

From the relation $\sum_j q_{ij} = v_i$, it is obtained that,

$$\sum_j q_{0j} = q_{02} = \mu = v_0$$

$$\sum_j q_{1j} = q_{10} + q_{13} = 3\lambda + \mu = v_1$$

$$\sum_j q_{2j} = q_{21} = \gamma = v_2$$

$$\sum_j q_{3j} = q_{32} = 3\lambda = v_3$$

This gives the equilibrium equations for the limiting probabilities $\boldsymbol{p} = \big(p_0, p_1, p_2, p_3\big)$:

$$\mu p_0 = 3\lambda p_1$$

$$(3\lambda + \mu)p_1 = \gamma p_2$$

$$\gamma p_2 = \mu p_0 + 3\lambda p_3$$

$$3\lambda p_3 = \mu p_1$$

c)

The desired probability, call it p_a, can be written as

$$p_a = P\big(X(t+h) = 1 \text{ or } 3 \,|\, X(t) = 0 \text{ or } 2\big)$$

$$= P\big(X(t+h) = 1 \,|\, X(t) = 0 \text{ or } 2\big) + P\big(X(t+h) = 3| \, X(t) = 0 \text{ or } 2\big)$$

$$= P\big(X(t+h) = 1 \,|\, X(t) = 0 \text{ or } 2\big)\,,$$

since obviously $P\big(X(t+h) = 3| \, X(t) = 0 \text{ or } 2\big) = 0$. Using that

$$P\big(A|B \cup C\big) = \frac{P\big(A \cap (B \cup C)\big)}{P\big(B \cup C\big)} = \frac{P\big(A|B\big)P(B) + P\big(A|C\big)P(C)}{P(B) + P(C)}\,,$$

we find that,

$$p_a = \frac{P\big(X(t+h) = 1 \,|\, X(t) = 2\big) P\big(X(t) = 2\big)}{P\big(X(t) = 0\big) + P\big(X(t) = 2\big)}\,,$$

since $P\big(X(t+h) = 1 \,|\, X(t) = 0\big) = 0$. Putting $P\big(X(t) = j\big) = p_j$ for large t, it follows that,

$$p_a = \frac{\gamma h p_2}{p_0 + p_2} + o(h)\,.$$

d)

It follows from the communication diagramme that this probability, call it p_b, is the same as the probability of going to 0 instead of 3 from 1. Hence

$$p_b = P\big(0 \text{ is visited before } 3 \;\,|\, X(0) = 3\big) = \frac{3\lambda}{\mu + 3\lambda}\,.$$

If $\mu \ll \lambda$, then $p_b \approx 0$, which implies that the MC moves primarily in the reduced diagramme:

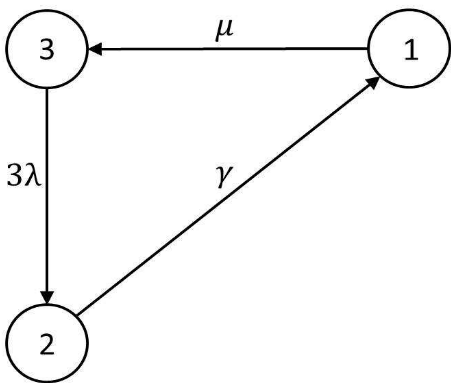

Figure 1: Reduced communication diagramme

This gives the reduced equilibrium equations:

$$\mu p_1 = \gamma p_2$$
$$\gamma p_2 = 3\lambda p_3$$
$$3\lambda p_3 = \mu p_1$$
$$p_1 + p_2 + p_3 = 1$$

which leads to the following solution:

$$p_1 = \frac{3\lambda\gamma}{3\lambda\gamma + 3\lambda\mu + \gamma\mu} = \frac{\gamma}{\mu} p_2 = \frac{3\lambda}{\mu} p_3\,.$$

Problem 2

a)

$a_0 = P_0$ due to Poison arrivals. Assuming that a customer pays 1 per unit of time while in service, the cost identity implies that 'average number in service $(= 1 - P_0) = \lambda\,\mathrm{E}[S]$. Hence,

$$a_0 = P_0 = 1 - \lambda\,\mathrm{E}[S]$$

where $a_0 =$ proportion of customers finding the system empty; $P_0 =$ proportion of time the system is empty; $\lambda =$ arrival rate.

b)

$$\mathrm{E}[S] = \mathrm{E}[S \mid 1.\text{ customer of busy period}]\,P[1.\text{ customer of busy period}]$$
$$+\mathrm{E}[S \mid \text{other than 1. customer of busy period}]\,P[\text{other than 1. customer of busy period}].$$
$$= \mathrm{E}[S_1]\,a_0 + \mathrm{E}[S_2]\,(1 - a_0)$$

c)

The relation

$$P_0 = \frac{\mathrm{E}[I]}{\mathrm{E}[I] + \mathrm{E}[B]} = \frac{\frac{1}{\lambda}}{\frac{1}{\lambda} + \mathrm{E}[B]}\,.$$

gives

$$\mathrm{E}[B] = \frac{1 - P_0}{\lambda P_0} = \frac{\mathrm{E}[S]}{1 - \lambda \mathrm{E}[S]}$$

From points a) and b) we find that

$$\mathrm{E}[S] = (1 - \lambda \mathrm{E}[S])\mathrm{E}[S_1] + \lambda \mathrm{E}[S]\,\mathrm{E}[S_2]\,,$$

which provides the equality,

$$\mathrm{E}[S] = \frac{\mathrm{E}[S_1]}{1 + \lambda \mathrm{E}[S_1] - \lambda \mathrm{E}[S_2]}$$

Substituting into the expression for $\mathrm{E}[B]$, it is obtained that

$$\mathrm{E}[B] = \frac{\mathrm{E}[S_1]}{1 - \lambda \mathrm{E}[S_2]}$$

Problem 3

a)

From the equation

$$P(B(t) \geq a) = P(B(t) \geq a | T_a \leq t) P(T_a \leq t) + P(B(t) \geq a | T_a > t) P(T_a > t)\,,$$

it is obtained that

$$P(B(t) \geq a) = P(B(t) \geq a | T_a \leq t) P(T_a \leq t)\,,$$

since clearly $P(B(t) \geq a | T_a > t) = 0$ ($T_a > t$ implies $B(t) < a$).

It is observed that $T_a \leq t$ ($t > 0$) implies that $B(s) = a$ for some $s \in (0, t)$ (with prob. 1). Due to symmetry, it follows that $P(B(t) \geq a | T_a \leq t) = P(B(t) \leq a | T_a \leq t) = 1/2$. This gives $P(T_a \leq t) = 2P(B(t) \geq a)$. Since $B(t) \sim N(0, t)$, it follows that the CDF $F_{T_a}(t) = P(T_a \leq t)$ is given as,

$$F_{T_a}(t) = 2P(B(t) \geq a) = 2P\Big(\frac{B(t)}{\sqrt{t}} \geq \frac{a}{\sqrt{t}}\Big) = 2\Big(1 - \Phi\Big(\frac{a}{\sqrt{t}}\Big)\,,$$

for $t > 0$, while $F_{T_a}(t) = 0$ for $t \leq 0$.

b)

Let $\hat{B}_t = \max_{0<s\leq t}\{B(s)\}$. Then $\{\hat{B}_t < a\} = \{T_a > t\}$.

Hence, it is obtained that

$$P(\hat{B}_t \leq a) = P(\hat{B}_t < a) = P(T_a > t) = 1 - P(T_a \leq t) = 1 - 2\Big(1 - \Phi\Big(\frac{a}{\sqrt{t}}\Big)\Big) = 2\Phi\Big(\frac{a}{\sqrt{t}}\Big) - 1\,.$$

The first equality follows since $B(t)$ is continuous (w.p. 1).

c)

$$P(T_a < \infty) = \lim_{t\to\infty} P(T_a \leq t) = 2\Big(1 - \lim_{t\to\infty} \Phi\Big(\frac{a}{\sqrt{t}}\Big)\Big) = 2\big(1 - \Phi(0)\big) = 2 \cdot \frac{1}{2} = 1\,.$$

$$\mathrm{E}[T_a] = \int_0^\infty t f_{T_a}(t)\, dt\,,$$

where (for $t > 0$),

$$f_{T_a}(t) = \frac{dF_{T_a}(t)}{dt} = 2\Big(1 - \frac{d}{dt}\Phi\Big(\frac{a}{\sqrt{t}}\Big)\Big) = 2\phi\Big(\frac{a}{\sqrt{t}}\Big)\frac{a}{2t^{3/2}} = \phi\Big(\frac{a}{\sqrt{t}}\Big)\frac{a}{t^{3/2}}\,.$$

Hence, it follows that,

$$\mathrm{E}[T_a] = a\int_0^\infty \phi\Big(\frac{a}{\sqrt{t}}\Big)\frac{dt}{\sqrt{t}} \geq a\int_1^\infty \phi\Big(\frac{a}{\sqrt{t}}\Big)\frac{dt}{\sqrt{t}} \geq a\phi(a)\int_1^\infty \frac{dt}{\sqrt{t}} = a\phi(a)\Big|_1^\infty 2\sqrt{t} = \infty\,.$$