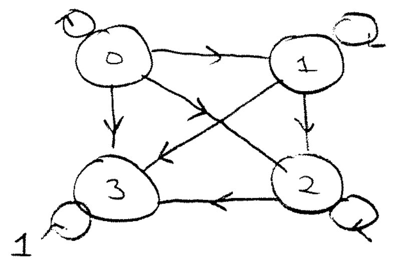
Oppgave 1:

a) Den stokastiske prosessen $X_0, X_1, X_2, \ldots$ vil være en Markov-kjede dersom den oppfyller Markov-egenskapen

$$P(X_{n+1}=j \mid X_n=i, X_{n-1}=k_{n-1}, \ldots, X_0) = P(X_{n+1}=j \mid X_n=i)$$

b)



4 klasser

$\{0\}, \{1\}, \{2\}$ transiente

$\{3\}$ rekurrent

$$P(X_2=3 \mid X_0=0) = \sum_{k=0}^{3} P(X_2=3 \mid X_0=0, X_1=k) \cdot P_{0k}$$

$$\overset{ME}{=} \sum_{k=0}^{3} P(X_2=3 \mid X_1=k) \cdot P_{0k} = \sum_{k=0}^{3} P_{0k} \cdot P_{k3}$$

$$= \frac{3}{5} \cdot \frac{1}{20} + \frac{1}{4} \cdot \frac{1}{10} + \frac{1}{10} \cdot \frac{3}{10} + \frac{1}{20} \cdot 1 = \frac{54}{4 \cdot 5 \cdot 20} = \underline{\underline{\frac{27}{200}}}$$

c) $T = \min\{n : X_n = 3\}$

$E[T \mid X_0 = i] = v_i \qquad i = 0, 1, 2$

$$v_0 = E[T \mid X_0 = 0] = \sum E[T \mid X_0 = 0, X_1 = k] \cdot P_{0k}$$

$$= (1+v_0)P_{00} + (1+v_1)P_{01} + (1+v_2)P_{02} + 1 \cdot P_{03}$$

$$= 1 + v_0 P_{00} + v_1 P_{01} + v_2 P_{02}$$

$$v_1 = 1 + v_1 P_{11} + v_2 P_{12}$$

$$v_2 = 1 + v_2 P_{22}$$

$$v_2 = \frac{1}{1-P_{22}} = \frac{1}{1-\frac{7}{10}} = \frac{1}{\frac{3}{10}} = \underline{\underline{\frac{10}{3}}}$$

$$v_1 = \frac{1 + v_2 P_{12}}{1 - P_{11}} = \frac{1 + \frac{10}{3} \cdot \frac{3}{10}}{1 - \frac{3}{5}} = \frac{2}{\frac{2}{5}} = \underline{\underline{5}}$$

$$v_0 = \frac{1 + v_1 P_{01} + v_2 P_{02}}{1 - P_{00}} = \frac{1 + 5 \cdot \frac{1}{4} + \frac{10}{3} \cdot \frac{1}{10}}{1 - \frac{3}{5}} = \underline{\underline{\frac{155}{24}}}$$

d) (stokk. mod) La S_{ij} være forventet tid i tilstand j gitt start i tilstand i

$$S_{ij} = E[\text{ tid i } j \mid \text{start i } i\,]$$
$$= \sum_k E[\text{ tid i } j \mid \text{start i } i, \text{ overgang til } k\,]\, P_{ik}$$
$$= \delta_{ij} + \sum_k P_{ik} S_{kj} \quad (\text{som gitt i formelsamling})$$

$$S_{00} = 1 + P_{00} S_{00} + P_{01} \overset{=0}{S_{10}} + P_{02} \overset{=0}{S_{20}}$$

$$S_{00} = \frac{1}{1-P_{00}} = \frac{1}{1-\frac{3}{5}} = \underline{\underline{\frac{5}{2}}}$$

$$S_{01} = P_{00} S_{01} + P_{01} S_{11} + P_{02} \overset{=0}{S_{21}}$$

$$S_{01} = \frac{P_{01} S_{11}}{1-P_{00}} \qquad S_{01} = \frac{\frac{1}{4}\cdot\frac{5}{2}}{\frac{2}{5}} = \underline{\underline{\frac{25}{16}}}$$

$$S_{11} = 1 + P_{11} S_{11} + P_{12} \overset{=0}{S_{21}}$$

$$S_{11} = \frac{1}{1-P_{11}} = \frac{1}{1-\frac{3}{5}} = \frac{5}{2}$$

$$S_{02} = P_{00} S_{02} + P_{01} S_{12} + P_{02} S_{22}$$

$$S_{22} = 1 + P_{22} S_{22} \qquad S_{22} = \frac{1}{1-P_{22}} = \frac{1}{1-\frac{7}{10}} = \frac{10}{3}$$

$$S_{12} = P_{11} S_{12} + P_{12} S_{22}$$

$$S_{12} = \frac{P_{12} S_{22}}{1-P_{11}} = \frac{\frac{3}{10}\cdot\frac{10}{3}}{1-\frac{3}{5}} = \frac{1}{\frac{2}{5}} = \frac{5}{2}$$

$$S_{02} = \frac{P_{01} S_{12} + P_{02} S_{22}}{1-P_{00}} = \frac{\frac{1}{4}\cdot\frac{5}{2} + \frac{1}{10}\cdot\frac{10}{3}}{1-\frac{3}{5}} = \underline{\underline{\frac{115}{48}}}$$

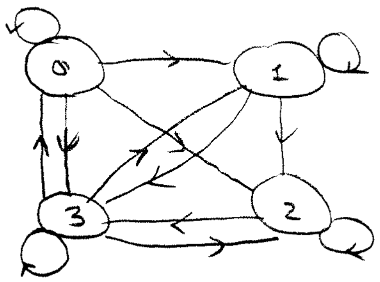
Kan kontrollere svaret fra d) ved at

$$v_2 = S_{22} = \frac{10}{3}$$

$$v_1 = S_{11} + S_{12} = \frac{5}{2} + \frac{5}{2} = 5$$

$$v_0 = S_{00} + S_{01} + S_{02} = \frac{5}{2} + \frac{25}{16} + \frac{115}{48} = \frac{310}{48} = \underline{\underline{\frac{155}{24}}}$$

e) Vi har fortsatt de 4 tilstandene



Vi trenger tilstand 3 for at vi skal kunne klassifisere maskinen defekt

Dersom vi kommer i tilstand 3 (defekt) blir ting reparert umiddelbart og maskinen oppfører seg da som om den ~~er~~ ok

$$P_{30} = P_{00} \quad P_{31} = P_{01} \quad P_{32} = P_{02} \quad P_{33} = P_{03}$$

En klasse $\{0,1,2,3\}$ rekurrent (positiv rekurrent)

Perioden for alle tilstander er 1 dvs aperiodisk

$$\pi_0 = \pi_0 P_{00} + \pi_1 \overset{=0}{P_{10}} + \pi_2 \overset{=0}{P_{20}} + \pi_3 P_{30}$$

$$(1-P_{00})\pi_0 = P_{00}\pi_3$$

$$\underline{\pi_3 = \frac{(1-P_{00})}{P_{00}}\pi_0}$$

$$\pi_1 = \pi_0 P_{01} + \pi_1 P_{11} + \pi_2 \overset{=0}{P_{21}} + \pi_3 P_{31}$$

$$(1-P_{11})\pi_1 = P_{01}(\pi_0 + \pi_3) \qquad (\pi_0 + \pi_3) = \frac{1}{P_{00}}\pi_0$$

$$\underline{\pi_1 = \frac{P_{01}}{P_{00}(1-P_{11})}\pi_0}$$

$$\pi_2 = \pi_0 P_{02} + \pi_1 P_{12} + \pi_2 P_{22} + \pi_3 P_{32}$$

$$(1-P_{22})\pi_2 = P_{02}(\pi_0 + \pi_3) + P_{12}\pi_1$$

$$= \frac{P_{02}}{P_{00}}\pi_0 + P_{12}\pi_1$$

$$\pi_2 = \frac{1}{1-P_{22}}\left[\frac{P_{02}}{P_{00}}\pi_0 + P_{12}\pi_1\right]$$

Innsatt gir

$$\pi_3 = \frac{2}{3}\pi_0 \quad , \quad \pi_1 = \frac{25}{24}\pi_0 \quad , \quad \pi_2 = \frac{115}{3\cdot 24}\pi_0$$

$$\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$$

$$\frac{3\cdot 24 + 25\cdot 3 + 115 + 2\cdot 24}{3\cdot 24}\pi_0 = 1$$

$$\pi_0 = \frac{3\cdot 24}{310} = \underline{\underline{\frac{72}{310}}} \qquad \pi_2 = \frac{115}{3\cdot 24}\cdot\frac{36}{155} = \underline{\underline{\frac{115}{310}}}$$

$$\pi_1 = \frac{25}{24}\,\frac{36}{155} = \underline{\underline{\frac{75}{310}}} \qquad \pi_3 = \frac{2}{3}\cdot\frac{36}{155} = \underline{\underline{\frac{48}{310}}}$$

f) $P(X_{n+1} = 3 \mid X_n \neq 3)$

(kun sloh. pros)

$$= \frac{P(X_{n+1} = 3 \cap X_n \neq 3)}{P(X_n \neq 3)}$$

$$= \frac{\sum_{k=0}^{2} P(X_{n+1} = 3 \mid X_n = k)\,P(X_n = k)}{\sum_{k=0}^{2} P(X_n = k)}$$

$$= \frac{\pi_0 P_{03} + \pi_1 P_{13} + \pi_2 P_{23}}{\pi_0 + \pi_1 + \pi_2}$$

Oppgave 2:

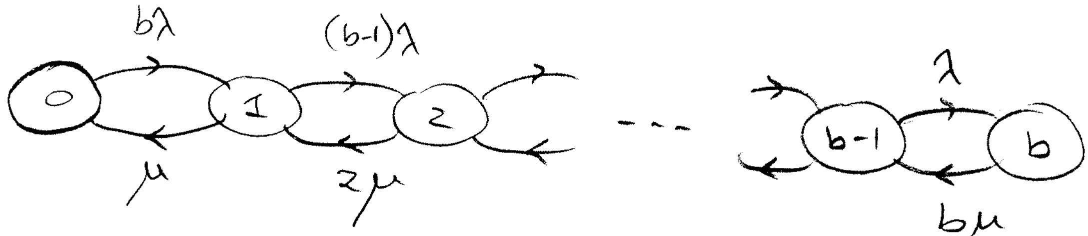
a) $N(t)$ stokastisk prosess med tilstander $0, 1, 2, \dots, b$

$N(t)$ er en kontinuerlig tid Markov-kjede siden

i) tiden i hver tilstand er eksponentialfordelt med forventning $1/v_i$

ii) når prosessen går fra tilstand i så går den til en annen tilstand j med sanns. P_{ij}

$N(t)$ er en fødsels- og dødsprosess siden man har en kont. tid Markov-kjede hvor man fra tilstand i enten går til tilstand $i-1$ (med rate μ_i) eller til tilstand $i+1$ (med rate λ_i)



Dersom 0 biler i reparasjon ($N(t)=0$) så er det b biler som kan feile uavh. av hverandre. Raten blir dermed $b\lambda$.

~~Dette~~ gir også ~~tettheten~~ for $T \sim$ eksponential $(b\lambda)$

$$P(T > t) = P(\text{ingen biler svikter i } (0,t])$$

$$= \prod_{k=1}^{b} P(\text{bil nr. } k \text{ svikter ikke i } (0,t])$$

$$= \prod_{k=1}^{b} e^{-\lambda t} = e^{-\lambda b t}$$

$$P(T < t) = \underline{\underline{1 - e^{-\lambda b t}}}$$

b) For fødsels- og dødsprosess

$$P_0 = \frac{1}{\sum_{k=0}^{b} \theta_k} \qquad P_k = \theta_k P_0$$

$$\theta_0 = 1 \quad \text{og} \quad \theta_k = \frac{\lambda_0 \lambda_1 \cdots \lambda_{k-1}}{\mu_1 \cdots \mu_k}$$

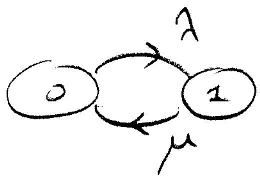
$$\theta_k = \frac{\lambda^k \, b(b-1)(b-2)\cdots(b-k+1)}{\mu^k \, 1\cdot 2\cdot 3 \cdots k} = \binom{b}{k}\left(\frac{\lambda}{\mu}\right)^k$$

$$P_0 = \frac{1}{\sum_{k=0}^{b} \binom{b}{k}\left(\frac{\lambda}{\mu}\right)^k} = \underline{\underline{\frac{1}{\left(1+\frac{\lambda}{\mu}\right)^b}}}$$

tatt ut

$$\left(\sum_{k=0}^{b} k \cdot P_k = \underline{\underline{\frac{1}{\left(1+\frac{\lambda}{\mu}\right)^b} \sum_{k=0}^{b} k \binom{b}{k}\left(\frac{\lambda}{\mu}\right)^k}}\right)$$

c)



Benytter forover-ligningene $P'_{ij}(t) = \sum_{k\neq j} P_{ik}(t) q_{kj} - P_{ij}(t) v_j$

$$P'_{00}(t) = P_{01}(t) q_{10} - P_{00}(t) v_0 \qquad \text{men} \quad \underline{P_{00}(t) + P_{01}(t) = 1}$$

$$= (1 - P_{00}(t))\mu - P_{00}(t)\lambda$$

$$P'_{00}(t) + (\lambda+\mu) P_{00}(t) = \mu \quad \text{og} \quad P_{00}(0) = 1$$

slik at

$$P_{00}(t) = e^{-(\lambda+\mu)t} + \mu \int_0^t e^{-(\lambda+\mu)(t-s)} ds$$

$$= e^{-(\lambda+\mu)t}\left[1 + \mu \int_0^t e^{+(\lambda+\mu)s} ds\right]$$

$$= e^{-(\lambda+\mu)t}\left[1 + \frac{\mu}{\lambda+\mu}\left(e^{(\lambda+\mu)t} - 1\right)\right]$$

$$= \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}$$

$$P_{01}(t) = 1 - P_{00}(t)$$

$$= \frac{\lambda}{\lambda+\mu} - \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}$$

$$P_0 = \lim_{t\to\infty} P_{00}(t) = \frac{\mu}{\lambda+\mu}$$

$$P_1 = \lim_{t\to\infty} P_{01}(t) = \frac{\lambda}{\lambda+\mu}$$

Oppgave 3:

a) Monte-Carlo simulering:

Stokastisk vektor $\mathbb{X} = (X_1, X_2, \dots, X_n)$ med tetthet $f(x_1, x_2, \dots, x_n)$

Ønsker $E[g(\mathbb{X})] = \int\int \dots \int g(\ )f(\ )\, dx_1\, dx_2 \dots dx_n$

Problem: Kan være vanskelig å løse analytisk/numerisk

Simuler $\mathbb{X}_1 = (X_{11}, X_{12}, \dots, X_{1n}) \dots \mathbb{X}_r$ alle uavh.

finn $\mathbb{Y}_1 = g(\mathbb{X}_1) \dots \mathbb{Y}_r = g(\mathbb{X}_r)$

Fra store talls lov har man at

$$\lim_{r \to \infty} \frac{\mathbb{Y}_1 + \mathbb{Y}_2 + \dots + \mathbb{Y}_n}{r} = E[\mathbb{Y}] = E[g(\mathbb{X})]$$

Inversjon:

Definer $X = F^{-1}(U)$ $\quad U \sim$ Uniform$(0,1)$

$$F_X(a) = P(X \leq a) = P(F^{-1}(U) \leq a)$$

og dessom F monoton

$$= P(U \leq F(a)) = F(a)$$

Simuler U og dann $X = F^{-1}(U)$

Da har X fordeling som F

$$u = F(x) = 1 - \exp(-\lambda x)$$

$$\exp(-\lambda x) = 1 - u$$

$$-\lambda x = \ln(1-u)$$

$$x = -\frac{1}{\lambda}\ln(1-u) \quad \text{eller } x = \frac{1}{\lambda}\ln u$$

$$\underline{X = -\frac{1}{\lambda}\ln U}$$

$X_i \sim \text{exponential}(\lambda)$

$Y = \sum_{i=1}^{n} X_i \sim \text{gamma}(n, \lambda)$

$= \sum_{i=1}^{n} -\frac{1}{\lambda} \ln U_i = \underline{-\frac{1}{\lambda} \sum_{i=1}^{n} \ln U_i} = \underline{-\frac{1}{\lambda} \ln(\prod_{i=1}^{n} U_i)}$

b) Rejection:

Har metode for å simulere fra $g(x)$
Ønsker å simulere fra $f(x)$

Anta det finnes en $c \geq \frac{f(y)}{g(y)} \quad \forall y$

Har da følgende algoritme:

1) simuler $Y \sim g(x)$ og $U \sim \text{uniform}(0,1)$
2) dersom $U \leq \frac{f(Y)}{cg(Y)}$ la $X = Y$
ellers returner punkt 1

$$P(X \leq x) = P(Y_N \leq x) = P\left(Y \leq x \mid U \leq \frac{f(Y)}{cg(Y)}\right)$$

$$= \frac{P(Y \leq x \cap U \leq f(Y)/cg(Y))}{K} \quad \text{hvor } K = \frac{f(Y)}{cg(Y)}$$

$$= \int P(Y \leq x \cap U \leq f(Y)/cg(Y) \mid Y = y) \cdot g(y)\,dy \,/ K$$

$$= \int_{-\infty}^{x} P(U \leq f(y)/cg(y)) \cdot g(y)\,dy \,/ K$$

$$= \int_{-\infty}^{x} \frac{f(y)}{cg(y)} \cdot g(y)\,dy / K = \underline{\frac{1}{cK} \int_{-\infty}^{x} f(y)\,dy}$$

$$P(X \leq \infty) = \frac{1}{cK} \cdot 1 = 1 \Rightarrow cK = 1$$

c) $g(x) \sim$ eksponentiell $\left(\frac{\lambda}{n}\right)$ $\quad \frac{\lambda}{n} e^{-\frac{\lambda}{n}x}$

$f(x) \sim$ gamma (n, λ) $\quad \frac{\lambda^n x^{n-1}}{(n-1)!} e^{-\lambda x}$

Bestemmer $c \geq \frac{f(x)}{g(x)}$:

$$h(x) = \frac{f(x)}{g(x)} = \frac{\lambda^n x^{n-1}}{(n-1)!} \frac{n}{\lambda} e^{\left(-\lambda x + \frac{\lambda}{n}x\right)}$$

$$= \frac{n \lambda^{n-1} x^{n-1}}{(n-1)!} e^{-\left(\frac{n-1}{n}\right)\lambda x}$$

$$\frac{\partial h(x)}{\partial x} = \frac{n\lambda^{n-1}}{(n-1)!}\left[(n-1)x^{n-2}e^{-\left(\frac{n-1}{n}\right)\lambda x} - \left(\frac{n-1}{n}\right)\lambda\, x^{n-1} e^{-\left(\frac{n-1}{n}\right)\lambda x}\right]$$

ekstremal for $\frac{\partial h(x)}{\partial x} = 0$

$$\cancel{(n-1)}\, \cancel{x^{n-2}} = \cancel{\frac{n-1}{n}}\lambda \cancel{x^{n-1}}\, x$$

$$\underline{x = \frac{n}{\lambda}}$$

$$c = \max h(x) = h\left(\frac{n}{\lambda}\right)$$

$$= \frac{n\lambda^{n-1}\left(\frac{n}{\lambda}\right)^{n-1}}{(n-1)!} e^{-\left(\frac{n-1}{n}\right)\lambda\frac{n}{\lambda}} = \underline{\frac{n^n}{(n-1)!} e^{-(n-1)}}$$

1. Simuler $Y \sim g(y)$ og $U \sim$ uniform$(0,1)$
2. Dersom

$$U \leq \frac{f(Y)}{c\,g(Y)} \quad \text{la } X = Y$$

$$\frac{f(Y)}{cg(Y)} = \frac{n\lambda^{n-1}}{\cancel{(n-1)!}} Y^{n-1} e^{-\left(\frac{n-1}{n}\right)\lambda Y} \cdot \frac{\cancel{(n-1)!}}{n^n} e^{(n-1)}$$

$$\ln U \leq \ln n + (n-1)\ln\lambda + (n-1)\ln Y - \left(\frac{n-1}{n}\right)\lambda Y$$
$$- n\ln n + (n-1)$$

$$= -(n-1)\ln n + (n-1)\ln\lambda + (n-1)\ln Y$$
$$- (n-1)\frac{\lambda}{n} Y + (n-1)$$
$$= (n-1)\left[\ln\left(\frac{\lambda}{n}Y\right) - \frac{\lambda}{n}Y + 1\right]$$

$Y \sim$ eksponensial $\left(\frac{\lambda}{n}\right)$

$Y_1 = -\ln U \quad \sim$ eksponensial (1)

$Y_2 = \frac{\lambda}{n} Y$

$$F_{Y_2}(y) = P(Y_2 \leq y) = P\left(\frac{\lambda}{n}Y \leq y\right) = P\left(Y \leq \frac{n}{\lambda}y\right)$$
$$= F_Y\left(\frac{n}{\lambda}y\right) = 1 - e^{-\frac{\lambda}{n}\cdot\frac{n}{\lambda}y} = 1 - e^{-y}$$

$Y_2 \sim$ eksponensial (1)

$$Y_1 \geq (n-1)\left[Y_2 - \ln Y_2 - 1\right]$$