Norges teknisk-naturvitenskapelige universitet
Institutt for matematiske fag

TMA4265 Stokastiske prosesser
Våren 2004

Løsningsforslag - Øving 5

Oppgaver fra læreboka

4.37

r paraplyer
$P\{Regn\} = p$

i) X_n = Antall paraplyer, <u>når han er hjemme</u>

$$P = \begin{bmatrix} 0 & 1 & 2 & ... & r-1 & r \\ (1-p) & p & 0 & 0 & 0 & 0 \\ p(1-p) & p^2+(1-p)^2 & (1-p)p & 0 & 0 & 0 \\ \ddots & \ddots & \ddots & & & \vdots \\ & & \ddots & \ddots & \ddots & (1-p)p \\ & & & & p(1-p) & p^2+(1-p)^2+(1-p)p \end{bmatrix}$$

ii) Irredusibel ergodisk MK, så grensefordelingen er gitt ved stasjonærfordelingen som oppfyller:

1.
$$\pi_k = \sum_{i=0}^{r} \pi_i P_{ik} \qquad , k = 0, .., r$$

2.
$$\sum_{i=0}^{r} \pi_i = 1$$

Dette gir ($q = 1 - p$)

1.

$$\begin{aligned}
\pi_0 &= \pi_0 q + \pi_1 pq \leftrightarrow \pi_0 = q\pi_1 \\
\pi_1 &= \pi_0 p + \pi_1(p^2+q^2) + \pi_2 pq \\
\pi_2 &= \pi_1 pq + \pi_2(p^2+q^2) + \pi_3 pq \\
&\vdots \\
\pi_r &= \pi_{r-1} pq + \pi_r(p^2+q^2+pq)
\end{aligned}$$

$\pi_1 = \ldots = \pi_r$ Sett inn og sjekk!

2.

$$\sum_{i=0}^{r} \pi_i = \pi_0 + \sum_{i=1}^{r} \pi_i = q\pi_1 + r\pi_1 = 1 \Rightarrow \pi_1 = \frac{1}{q+r}$$

Dermed

$$\pi_1 = ... = \pi_r = \underline{\underline{\frac{1}{q+r}}}$$

$$\pi_0 = q\pi_r = \underline{\underline{\frac{q}{q+r}}}$$

iii)

$$\begin{aligned}
P_r\{V\text{åt}\} &= P_r\{V\text{åt} \mid \textit{Starter hjemme}\} \cdot P_r\{\textit{Starter hjemme}\} \\
&\quad + P_r\{V\text{åt} \mid \textit{Jobb}\} \cdot P_r\{\textit{Jobb}\} \\
&= P_r\{\textit{Regn} \cap 0 \textit{ paraply. hjemme} \mid \textit{Hj.}\} \cdot P_r\{\textit{Hj}\} \\
&\quad + P_r\{\textit{Regn} \cap 0 \text{ parapl.på jobb}\} \cdot P_r\{\textit{Jobb}\} \\
&= p \cdot \pi_0 \cdot \frac{1}{2} + p \cdot [\pi_r \cdot q] \cdot \frac{1}{2}
\end{aligned}$$

$$= \underline{\underline{\frac{pq}{q+r}}}$$

iv)
For $r = 3$, sett
$f(p) = P_r\{Vaat\} = \frac{p(1-p)}{1-p-3} = \frac{p-p^2}{4-p}$
$f'(p) = \frac{p^2-8p+4}{(4-p)^2} = 0\underline{\underline{p = 4 - 2\sqrt{3}}} = 0,54$

4.52

Finn
- $f_{i3} = P_r\{X_n = 3 \text{ for } n > 1 \mid X_0 = i\}$ (s.201)
- s_{i3} = Forventet tid i 3 | $X_0 = i$ (s.200)

Fra s.200 har vi at
$s_{ij} = \delta_{ij} + \sum_k P_{ik} s_{kj}$ (*)

Det holder å summere over de transiente tilstandene, siden $s_{ik} = 0$ når k er absorberende.
Vi har at

$$P = \begin{bmatrix} 0,4 & 0,2 & 0,1 & 0,3 \\ 0,1 & 0,5 & 0,2 & 0,2 \\ 0,3 & 0,4 & 0,2 & 0,1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Fra (*) får vi at

$$\underline{s} = I + P_T \underline{s} \Rightarrow \underline{s} = (I - P_T)^{-1} \text{når } \underline{s} = \begin{bmatrix} s_{11} & \dots & s_{13} \\ s_{21} & \dots & s_{23} \\ s_{31} & \dots & s_{33} \end{bmatrix}$$

Setter inn og får

$$\underline{s} = (1 - P_T)^{-1} = \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \begin{bmatrix} 2,2 & 1,3 & 0,62 \\ 0,96 & 3,1 & 0,89 \\ 1,3 & 2,0 & 1,9 \end{bmatrix}$$

Fra s.202 har vi at

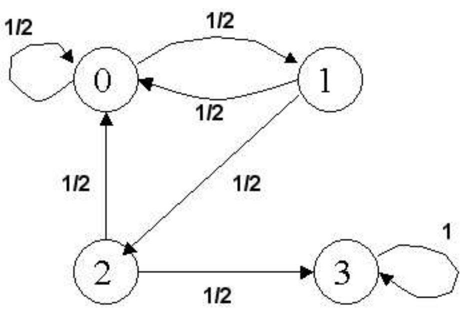
$$f_{ij} == \frac{s_{ij} - \delta_{ij}}{s_{jj}}$$

$$\Downarrow$$

$$\begin{aligned} f_{13} &= \underline{\underline{0,33}} \\ f_{23} &= \underline{\underline{0,47}} \\ f_{33} &= \underline{\underline{0,47}} \end{aligned}$$

Eksamensoppgaver

Aug. '98, oppg.1



a)

$$\begin{aligned} Pr\{X_2 = 1 | X_0 = 0\} &= Pr\{X_1 = 0 \cap X_2 = 1 | X_0 = 0\} \\ &= Pr\{X_1 = 0 | X_0 = 0\} \cdot Pr\{X_2 = 1 | X_1 = 0\} \\ &= \frac{1}{2} \cdot \frac{1}{2} = \underline{\underline{\frac{1}{4}}} \end{aligned}$$

$$\begin{aligned} Pr\{X_3 = 0 | X_0 = 0\} &= Pr\{X_1 = 0 \cap X_2 = 0 \cap X_3 = 0 | X_0 = 0\} \\ &+ Pr\{X_1 = 0 \cap X_2 = 1 \cap X_3 = 0 | X_0 = 0\} \\ &+ Pr\{X_1 = 1 \cap X_2 = 0 \cap X_3 = 0 | X_0 = 0\} \\ &+ Pr\{X_1 = 1 \cap X_2 = 2 \cap X_3 = 0 | X_0 = 0\} \\ &= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \underline{\underline{\frac{1}{2}}} \end{aligned}$$

$$\begin{aligned} Pr\{X_4 = 3 | X_0 = 0\} &= Pr\{X_1 = 0 \cap X_2 = 1 \cap X_3 = 2 \cap X_4 = 3 | X_0 = 0\} \\ &+ Pr\{X_1 = 1 \cap X_2 = 2 \cap X_3 = 3 \cap X_4 = 3 | X_0 = 0\} \\ &= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot 1 = \underline{\underline{\frac{3}{16} = 0,1875}} \end{aligned}$$

b) La $v_i = E[T|X_0 = i], T = \min_{n\geq 0}\{n : X_n = 3\}$

Førstestegsanalyse for da ligningene

$$\begin{aligned}
v_0 &= \frac{1}{2}v_0 + \frac{1}{2}v_1 + 1\\
v_1 &= \frac{1}{2}v_0 + \frac{1}{2}v_2 + 1\\
v_2 &= \frac{1}{2}v_0 + \frac{1}{2}v_3 + 1\\
v_3 &= 0
\end{aligned}$$

som gir

$$\begin{aligned}
v_2 &= \frac{1}{2}v_0 + 1\\
v_1 &= \frac{1}{2}v_0 + \frac{1}{2}(\frac{1}{2}v_0 + 1) + 1 = \frac{3}{4}v_0 + \frac{3}{2}\\
v_0 &= \frac{1}{2}v_0 + \frac{1}{2}(\frac{3}{4}v_0 + \frac{3}{2}) + 1 = \frac{7}{8}v_0 + \frac{7}{4}
\end{aligned}$$

$\Rightarrow v_0 = \frac{7}{4} \cdot 8 = \underline{\underline{14}}$

c) La Y_n være antall av de siste kastene som stemmer med starten av sekvensen MKK og får overgangsmatrisen

$$\begin{array}{c} \\ 0\\1\\2\\3 \end{array}\begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array}\\ \begin{pmatrix} 1/2 & 1/2 & 0 & 0\\ 0 & 1/2 & 1/2 & 0\\ 0 & 1/2 & 0 & 1/2\\ 0 & 0 & 0 & 1 \end{pmatrix}\end{array}$$

Førstestegsanalyse gir nå ligningene:

$$\begin{aligned}
v_0 &= \frac{1}{2}v_0 + \frac{1}{2}v_1 + 1\\
v_1 &= \frac{1}{2}v_1 + \frac{1}{2}v_2 + 1\\
v_2 &= \frac{1}{2}v_1 + \frac{1}{2}v_3 + 1\\
v_3 &= 0
\end{aligned}$$

som gir

$$
\begin{aligned}
v_2 &= \frac{1}{2}v_1 + 1 \\
v_1 &= \frac{1}{2}v_1 + \frac{1}{2}(\frac{1}{2}v_1 + 1) + 1 = \frac{3}{4}v_1 + \frac{3}{2} \\
&\Rightarrow v_1 = \frac{3}{2}\cdot 4 = 6 \\
v_0 &= \frac{1}{2}v_0 + \frac{1}{2}\cdot 6 + 1 \\
&\Rightarrow v_0 = 4\cdot 2 = \underline{\underline{8}}
\end{aligned}
$$

Rimelig at dette svaret blir mindre enn i b) siden i kjeden X_n faller man helt tilbake til 0 dersom man får et “feil kast”, mens i kjeden Y_n faller man bare tilbake til 1.

d) Ligningssystem for stasjonærfordeling

$$
\begin{aligned}
\pi_0 &= \frac{1}{2}\pi_0 + \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2 + \frac{1}{2}\pi_3 \\
\pi_1 &= \frac{1}{2}\pi_0 \\
\pi_2 &= \frac{1}{2}\pi_1 \\
\pi_0 &+ \pi_1 + \pi_2 + \pi_3 = 1
\end{aligned}
$$

slik at

$$\pi_0 = \frac{1}{2}\pi_0 + \frac{1}{4}\pi_0 + \frac{1}{8}\pi_0 + \frac{1}{2}\pi_3$$

$$\Rightarrow \frac{1}{8}\pi_0 = \frac{1}{2}\pi_3 \Rightarrow \pi_3 = \frac{1}{4}\pi_0$$

og

$$\pi_0 + \frac{1}{2}\pi_0 + \frac{1}{4}\pi_0 + \frac{1}{4}\pi_0 = 2\pi_0 = 1 \Rightarrow \pi_0 = \frac{1}{2}$$

$$\underline{\underline{\pi_0 = \frac{1}{2}, \pi_1 = \frac{1}{4}, \pi_2 = \frac{1}{8}, \pi_3 = \frac{1}{8}}}$$

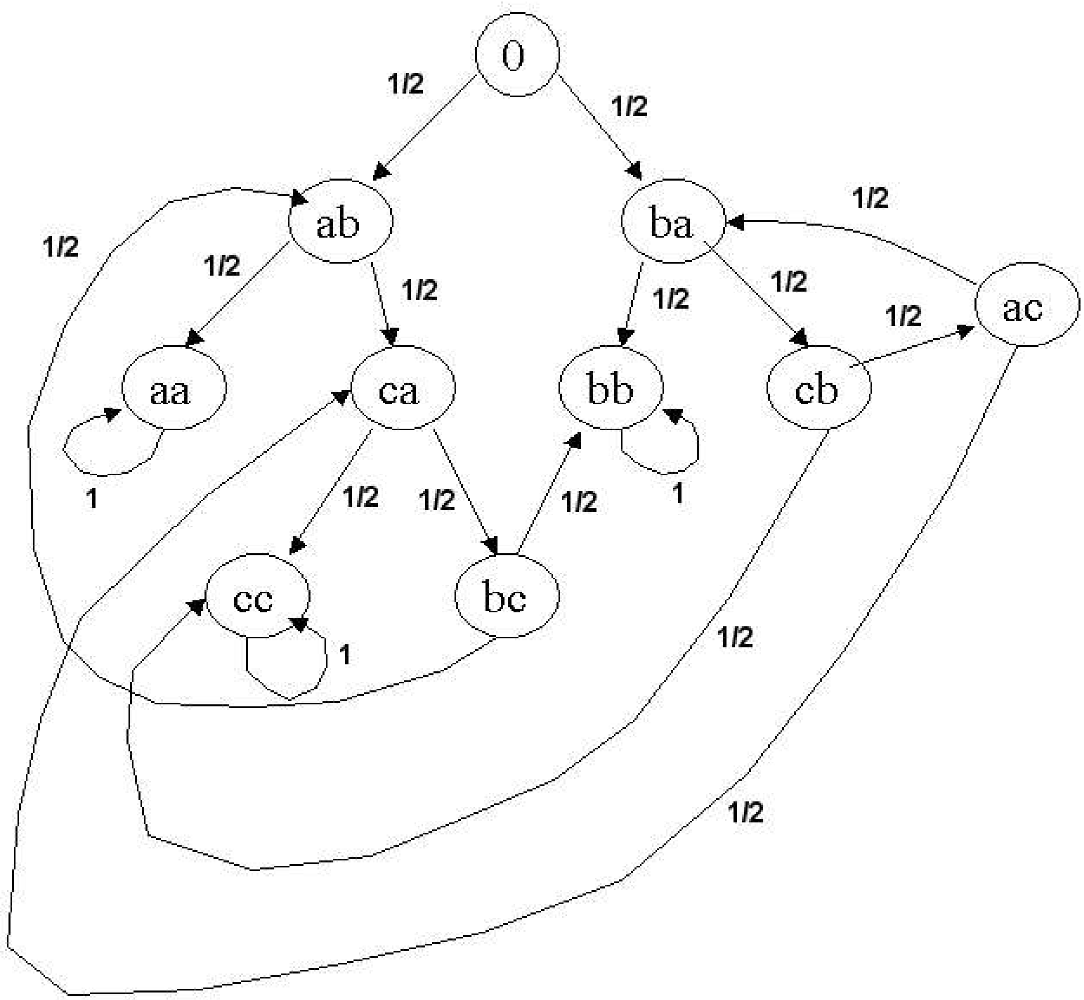
Kjeden har en grensefordeling fordi den er irredusibel og aperiodisk ($P_{00} > 0$) . Grensefordelingen er lik stasjonærfordelingen over.

I det lange løp vil man etter $\frac{1}{8}$ av kastene ha MMM som tre siste kast.

Jan. '99, oppg.5

$$X_n = \begin{array}{lll} 0 & ; & \text{ingen kamp spilt} \\ ab & ; & \text{forrige kamp var mellom a og b og a vant} \\ ac & ; & \text{forrige kamp var mellom a og c og a vant} \\ ba & & \\ bc & & \\ ca & & \\ cb & & \\ aa & ; & \text{a er turnerings vinner} \\ bb & ; & \\ cc & ; & \end{array}$$

overgangssannsynligheter:



La $T = min\{n \geqslant 0 : X_n = aa \;\; X_n = bb \,\text{eller} X_n = cc\}$
$\mu_i = P_r\{X_T = aa \mid X_0 = i\}$

Får ligningene:

$$
\begin{aligned}
\mu_0 &= \frac{1}{2}\mu_{ab} + \frac{1}{2}\mu_{ba} \\
\mu_{ab} &= \frac{1}{2} + \frac{1}{2}\mu_{ca} \\
\mu_{ba} &= \frac{1}{2}\mu_{cb} \\
\mu_{ca} &= \frac{1}{2}\mu_{bc} \\
\mu_{cb} &= \frac{1}{2}\mu_{ca} \\
\mu_{ac} &= \frac{1}{2} + \frac{1}{2}\mu_{ba} \\
\mu_{bc} &= \frac{1}{2}\mu_{ab}
\end{aligned}
$$

Symmetrier gir at

$$
\begin{aligned}
\mu_{ab} &= \mu_{ac} \\
\mu_{ba} &= \mu_{ca} \\
\mu_{bc} &= \mu_{cb}
\end{aligned}
$$

dvs.

$$
\begin{aligned}
\mu_0 &= \frac{1}{2}\mu_{ab} + \frac{1}{2}\mu_{ba} \\
\mu_{ba} &= \frac{1}{2}\mu_{bc} \\
\mu_{ab} &= \frac{1}{2} + \frac{1}{2}\mu_{ba} \\
\mu_{bc} &= \frac{1}{2}\mu_{ab}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \mu_{ab} &= \frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\mu_{ab} \\
\Rightarrow \mu_{ab} &= \frac{\frac{1}{2}}{1-\frac{1}{8}} \\
\mu_{ab} &= \frac{4}{8-1} = \underline{\underline{\frac{4}{7}}} \\
\Rightarrow \mu_{ba} &= \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{4}{7} = \underline{\frac{1}{7}}
\end{aligned}
$$

dvs.

$$\mu_0 = \frac{1}{2}\cdot\frac{4}{7} + \frac{1}{2}\cdot\frac{1}{7} = \underline{\underline{\frac{5}{14}}} = 0.357$$

Aug. '99, oppg.1

a)

$$\left[\begin{array}{ccc|ccc|ccc}
 & \frac{1}{2} & & \frac{1}{2} & & & & & \\
\frac{1}{3} & & \frac{1}{3} & & \frac{1}{3} & & & & \\
 & \frac{1}{2} & & & & \frac{1}{2} & & & \\
\frac{1}{3} & & & & \frac{1}{3} & & \frac{1}{3} & & \\
 & \frac{1}{4} & & \frac{1}{4} & & \frac{1}{4} & & \frac{1}{4} & \\
 & & \frac{1}{3} & & \frac{1}{3} & & & & \frac{1}{3} \\
 & & & \frac{1}{2} & & & & \frac{1}{2} & \\
 & & & & \frac{1}{3} & & \frac{1}{3} & & \frac{1}{3} \\
 & & & & & \frac{1}{2} & & \frac{1}{2} &
\end{array}\right]$$

$$\begin{aligned}
P_{15}^{(2)} &= P_r\{X_2 = 5 \mid X_0 = 1\} \\
&= P_{12} \cdot P_{25} + P_{14} \cdot P_{45} = \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} = \underline{\underline{\frac{1}{3}}}
\end{aligned}$$

$$\begin{aligned}
P_r\{X_2 = 5 \mid X_0 = 1 \cap X_3 = 2\} &= \frac{P_r\{X_2 = 5 \cap X_3 = 2 \mid X_0 = 1\}}{P_r\{X_3 = 2 \mid X_0 = 1\}} \\
&= \frac{P_r\{X_3 = 2 \mid X_2 = 5 \cap X_0 = 1\} \cdot P_r\{X_2 = 5 \mid X_0 = 1\}}{P_r\{X_3 = 2 \mid X_0 = 1\}} \\
&= \frac{P_r\{X_3 = 2 \mid X_2 = 5\} \cdot P_r\{X_2 = 5 \mid X_0 = 1\}}{P_r\{X_3 = 2 \mid X_0 = 1\}} = \frac{P_{52} \cdot P_{15}^{(2)}}{P_{12}^{(3)}}
\end{aligned}$$

$$\begin{aligned}
P_{12}^{(3)} &= P_{12} \cdot P_{23} \cdot P_{32} \cdot + P_{12} \cdot P_{25} \cdot P_{52} \cdot + P_{14} \cdot P_{42} \cdot P_{52} \cdot + P_{14} \cdot P_{41} \cdot P_{12} \cdot \\
&= \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{4}
\end{aligned}$$

$$P_r\{X_2 = 1 \mid \quad X_3 = 2\} = \frac{\frac{1}{4} \cdot \frac{1}{3}}{\frac{1}{4}} = \underline{\underline{\frac{1}{2}}}$$

b) La $T = min\{n \geqslant 0 : X_n = 1\}$: antall tidssteg til musa er i kupe nr. 9 for første gang.
La

$$v_i = E[T \mid X_0 = i] \tag{1}$$

Fra symmetrien i labyrinten ser man at:

$$\begin{aligned}
v_2 &= v_4 \\
v_3 &= v_7 \\
v_6 &= v_8
\end{aligned}$$

Har dessuten at $v_9 = 0$.
Førstestegsanalyse gir:

$$
\begin{aligned}
v_1 &= 1 + \frac{1}{2}v_2 + \frac{1}{2}v_4 = 1 + v_2 \\
v_2 &= 1 + \frac{1}{3}v_3 + \frac{1}{3}v_5 \\
v_3 &= 1 + \frac{1}{2}v_2 + \frac{1}{2}v_6 \\
v_5 &= 1 + \frac{1}{4}v_2 + \frac{1}{4}v_4 + \frac{1}{4}v_6 + \frac{1}{4}v_8 = 1 + \frac{1}{2}v_2 + \frac{1}{2}v_6 \\
v_6 &= 1 + \frac{1}{3}v_3 + \frac{1}{3}v_5 + +\frac{1}{3} \cdot 0
\end{aligned}
$$

Dvs. ligningssystem med 5 ukjente. Løsning av ligningssystemet er:
$v_1 = 18, v_2 = 17, v_3 = 15, v_5 = 15, v_6 = 11$

dvs.

$$E[T \mid X_0 = 1] = v_1 = \underline{\underline{18}}$$

c) Fra symmetrien i labyrinten ser en at

$$
\begin{aligned}
\Pi_1 &= \Pi_3 = \Pi_7 = \Pi_9 \\
\Pi_2 &= \Pi_4 = \Pi_6 = \Pi_8
\end{aligned}
$$

Dermed nok med tre ligninger (Π_1, Π_2, Π_5)

$$
\begin{aligned}
\Pi_1 &= \frac{1}{3}\Pi_2 + \frac{1}{3}\Pi_4 = \frac{2}{3}\Pi_2 \\
\Pi_1 &= \frac{1}{2}\Pi_1 + \frac{1}{2}\Pi_3 = \frac{1}{4}\Pi_5 = \Pi_1 + \frac{1}{4}\Pi_5 \\
4\Pi_1 + 4\Pi_2 + \Pi_5 = 1
\end{aligned}
$$

Løsning av ligningssystemet er:
$\Pi_1 = \frac{1}{12}, \Pi_2 = \frac{1}{8}, \Pi_5 = \frac{1}{6}$

Dvs. vil være i kupe 1,3,7 og 9 i $\frac{1}{12}$ av tiden.
Dvs. vil være i kupe 2,4,6 og 8 i $\frac{1}{8}$av tiden.
Dvs. vil være i kupe 5 i $\frac{1}{6}$av tiden.

La m være tidspunktet vi kommer tilbake og la V vare antall tidssteg til musa når kupe nr. 9

$$\begin{aligned}
E[V] &= \sum_{i=1}^{9} E[V \mid X_m = i] \cdot P_r\{X_m = i\} \\
&= \sum_{i=1}^{9} v_i \cdot \Pi_i \\
&= 18 \cdot \frac{1}{12} + 17 \cdot \frac{1}{8} + 15 \cdot \frac{1}{12} + 17 \cdot \frac{1}{8} \\
&\quad + 15 \cdot \frac{1}{6} + 11 \cdot \frac{1}{8} + 15 \cdot \frac{1}{12} + 11 \cdot \frac{1}{8} + 0 \cdot \frac{1}{12} \\
&= \underline{\underline{\frac{27}{2} = 13.5}}
\end{aligned}$$