TMA4285 Tidsrekker og filterteori

Norges teknisk-naturvitenskapelige universitet
Institutt for matematiske fag

Løsningsforslag - Øving 1 august 2004

Exercise 1.1

a)

Let $S(c) = E[(Y-c)^2]$. Then

$$S(c) = E(Y^2) - 2cE(Y) + c^2$$

This gives

$$\frac{dS}{dc} = -2E(Y) + 2c = 0$$

for $c = E(Y)$, which leads to a global minimum since $\frac{d^2S}{dc^2} = 2 > 0$ for all c.

b)

$$\begin{aligned}
E[\big(Y-f(X)\big)^2|X] &= E[\big(Y-E(Y|X)+E(Y|X)-f(X)\big)^2|X] = \\
E[\big(Y-E(Y|X)\big)^2|X] &+ 2E[\big(Y-E(Y|X)\big)\big(E(Y|X)-f(X)\big)|X] + E[\big(E(Y|X)-f(X)\big)^2|X] = \\
E[\big(Y-E(Y|X)\big)^2|X] &+ 2\big(E(Y|X)-f(X)\big)E[\big(Y-E(Y|X)\big)|X] + E[\big(E(Y|X)-f(X)\big)^2|X] = \\
E[\big(Y-E(Y|X)\big)^2|X] &+ E[\big(E(Y|X)-f(X)\big)^2|X] \geq E[\big(Y-E(Y|X)\big)^2|X]
\end{aligned}$$

because $E(Y|X)$ is a function of X and $E(g(X)Y|X) = g(X)E(Y|X)$ for any function g such that $E(g(X)Y)$ exists.

It follows that

$$E[\big(Y-E(Y|X)\big)^2|X] \leq E[\big(Y-f(X)\big)^2|X]$$

for any function f. Hence $E[\big(Y-f(X)\big)^2|X]$ is minimized when $f(X) = E(Y|X)$.

c)

Since

$$E[(Y-E(Y|X))^2] = E\Big(E[(Y-E(Y|X))^2|X]\Big) \leq E\Big(E[(Y-f(X))^2|X]\Big) = E[(Y-f(X))^2]$$

it follows immediately that the random variable $f(X)$ that minimizes $E[(Y-f(X))^2]$ is $f(X) = E(Y|X)$.

Exercise 1.2

a)

Let $X = (X_1, X_2, \ldots, X_n)$. Then

$$E[(X_{n+1} - f(X))^2|X] = E[(X_{n+1} - E(X_{n+1}|X) + E(X_{n+1}|X) - f(X))^2|X] =$$
$$E[(X_{n+1} - E(X_{n+1}|X))^2|X] + 2E[(X_{n+1} - E(X_{n+1}|X))(E(X_{n+1}|X) - f(X))|X]$$
$$+ E[(E(X_{n+1}|X) - f(X))^2|X] =$$
$$E[(X_{n+1} - E(X_{n+1}|X))^2|X] + 2(E(X_{n+1}|X) - f(X))E[(X_{n+1} - E(X_{n+1}|X))|X]$$
$$+ E[(E(X_{n+1}|X) - f(X))^2|X] =$$
$$E[(X_{n+1} - E(X_{n+1}|X))^2|X] + E[(E(X_{n+1}|X) - f(X))^2|X] \geq E[(X_{n+1} - E(X_{n+1}|X))^2|X]$$

because $E(X_{n+1}|X)$ is a function of X and $E(g(X)X_{n+1}|X) = g(X)E(X_{n+1}|X)$ for any function g such that $E(g(X)X_{n+1})$ exists.

It follows that

$$E[(X_{n+1} - E(X_{n+1}|X))^2|X] \leq E[(X_{n+1} - f(X))^2|X]$$

for any function f. Hence $E[(X_{n+1}-E(X_{n+1}|X))^2|X]$ is minimized when $f(X) = E(X_{n+1}|X)$.

b)

Since

$$E[(X_{n+1} - E(X_{n+1}|X))^2] = E(E[(X_{n+1} - E(X_{n+1}|X))^2|X])$$
$$\leq E(E[(X_{n+1} - f(X))^2|X]) = E[(X_{n+1} - f(X))^2]$$

it follows immediately that the random variable $f(X)$ that minimizes $E[\big(X_{n+1} - f(X)\big)^2]$ is again $f(X) = E(X_{n+1}|X)$.

c)

By b) the minimum mean-squared error predictor of X_{n+1} in terms of $X = (X_1, X_2, \ldots, X_n)$ when $X_t \sim IID(\mu, \sigma^2)$ is

$$E(X_{n+1}|X) = E(X_{n+1}) = \mu$$

d)

Suppose that $\sum_{i=1}^n \alpha_i X_i$ is an unbiased estimator for μ, that is, $\sum_{i=1}^n \alpha_i = 1$. Then

$$E[\big(\sum_{i=1}^n \alpha_i X_i - \mu\big)^2] = E[\big(\sum_{i=1}^n \alpha_i X_i - \overline{X}\big)^2] + 2E[\big(\sum_{i=1}^n \alpha_i X_i - \overline{X}\big)\big(\overline{X} - \mu\big)] + E[\big(\overline{X} - \mu\big)^2] \geq E[\big(\overline{X} - \mu\big)^2]$$

since the second term is zero: $E[\big(\sum_{i=1}^n \alpha_i X_i - \overline{X}\big)\big(\overline{X} - \mu\big)] = Cov(\sum_{i=1}^n \alpha_i X_i - \overline{X}, \overline{X}) = Cov(\sum_{i=1}^n \alpha_i X_i, \sum_{i=1}^n \frac{1}{n} X_i) - Cov(\sum_{i=1}^n \frac{1}{n} X_i, \sum_{i=1}^n \frac{1}{n} X_i) = \sum_{i=1}^n \frac{\alpha_i}{n}\sigma^2 - \sum_{i=1}^n \frac{1}{n^2}\sigma^2 = 0$.

e)

Again, suppose that $\sum_{i=1}^n \alpha_i X_i$ is an unbiased estimator for μ, that is, $\sum_{i=1}^n \alpha_i = 1$. Then

$$E[\big(X_{n+1} - \sum_{i=1}^n \alpha_i X_i\big)^2] = E[\big(X_{n+1} - \overline{X}\big)^2] + 2E[\big(X_{n+1} - \overline{X}\big)\big(\overline{X} - \sum_{i=1}^n \alpha_i X_i\big)] + E[\big(\overline{X} - \sum_{i=1}^n \alpha_i X_i\big)^2]$$
$$\geq E[\big(X_{n+1} - \overline{X}\big)^2]$$

since the second term is zero: $Cov\big(X_{n+1} - \overline{X}, \overline{X} - \sum_{i=1}^n \alpha_i X_i\big) = -Cov(\overline{X}, \overline{X}) + Cov(\overline{X}, \sum_{i=1}^n \alpha_i X_i) = 0$ as in d).

f)

$$E\big(S_{n+1}|S_1, \ldots, S_n\big) = E\big(S_n + X_{n+1}|S_1, \ldots, S_n\big) = S_n + E\big(X_{n+1}|S_1, \ldots, S_n\big) = S_n + \mu$$

since X_{n+1} is independent of $S_1, \ldots, S_n$.

Exercise 1.3

i)

$E(X_t)$ is independent of t since the distribution of X_t is independent of t and $E(X_t)$ exists.

ii)

Since $E[X_{t+h}X_t]^2 \leq E[X_{t+h}^2]E[X_t^2]$ for all integers t, h, and the joint distribution of X_{t+h} and X_t is independent of t, it follows that $E[X_{t+h}X_t]$ exists and is independent of t for every integer h.

Combining i) and ii) it follows that X_t is weakly stationary.

Exercise 1.4

a)

$E(X_t) = a$ is independent of t.

$$Cov(X_{t+h}, X_t) = \begin{cases} (b^2 + c^2)\sigma^2 & ; \quad h = 0 \\ 0 & ; \quad h = \pm 1 \\ bc\sigma^2 & ; \quad h = \pm 2 \\ 0 & ; \quad |h| > 2 \end{cases}$$

which is independent of t. That is, X_t is stationary.

b)

$E(X_t) = 0$ is independent of t.

$$Cov(X_{t+h}, X_t) = Cov(Z_1 \cos c(t+h) + Z_2 \sin c(t+h), Z_1 \cos ct + Z_2 \sin ct)$$
$$= \sigma^2 \big(\cos c(t+h) \cos ct + \sin c(t+h) \sin ct \big) = \sigma^2 \cos ch$$

which is independent of t. That is, X_t is stationary.

c)

$E(X_t) = 0$ is independent of t.

$$Cov(X_{t+1}, X_t) = \sigma^2 \cos c(t+1) \sin ct$$

which is not independent of t. That is, X_t is not stationary (except in the special case when c is an integer multiple of 2π).

d)

$E(X_t) = a$ is independent of t.

$$Cov(X_{t+h}, X_t) = b^2\sigma^2$$

which is independent of t. That is, X_t is stationary.

e)

$E(X_t) = 0$ is independent of t.

$$Cov(X_{t+h}, X_t) = \sigma^2 \cos c(t+h) \cos ct$$

which is not independent of t. That is, X_t is not stationary (except in the special case when c is an integer multiple of 2π).

f)

$E(X_t) = 0$ is independent of t.

$$Cov(X_{t+h}, X_t) = E[X_{t+h}X_t] = E[Z_{t+h}Z_{t+h-1}Z_tZ_{t-1}] = \begin{cases} \sigma^4 & ; \quad h = 0 \\ 0 & ; \quad |h| > 0 \end{cases}$$

which is independent of t. That is, X_t is stationary, and it is seen that in fact $X_t \sim WN(0, \sigma^4)$.

Exercise 1.5

a)

The autocovariance function

$$\gamma_X(h) = \begin{cases} 1+\theta^2 & ; \quad h = 0 \\ \theta & ; \quad h = \pm 2 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

The autocorrelation function

$$\rho_X(h) = \begin{cases} 1 & ; \quad h = 0 \\ \frac{\theta}{1+\theta^2} & ; \quad h = \pm 2 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

For $\theta = 0.8$ it is obtained that

$$\gamma_X(h) = \begin{cases} 1.64 & ; \quad h = 0 \\ 0.8 & ; \quad h = \pm 2 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

$$\rho_X(h) = \begin{cases} 1 & ; \quad h = 0 \\ 0.488 & ; \quad h = \pm 2 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

b)
Let $\overline{X}_4 = \frac{1}{4}(X_1 + \ldots + X_4)$. Then

$$Var(\overline{X}_4) = Cov\big(\overline{X}_4, \overline{X}_4\big) = \frac{1}{16}\sum_{i=1}^{4}\sum_{i=1}^{4} Cov\big(X_i, X_j\big)$$
$$= \frac{1}{4}\big(\gamma_X(0) + \gamma_X(2)\big) = \frac{1}{4}\big(1.64 + 0.8\big) = 0.61$$

c)

$$Var(\overline{X}_4) = Cov\big(\overline{X}_4, \overline{X}_4\big) = \frac{1}{4}\big(\gamma_X(0) + \gamma_X(2)\big) = \frac{1}{4}\big(1.64 - 0.8\big) = 0.21$$

The negative lag 2 correlation in c) means that positive deviations of X_t from zero tend to be followed two time units later by a compensating negative deviation, resulting in smaller variability in the sample mean than in b) (and also smaller than if the time series X_t were IID(0, 1.64) in which case $Var(\overline{X}_4) = 0.41$).