Løsningsforslag - Øving 2 september 2004

Exercise 1.7

$E(X_t + Y_t) = \mu_X + \mu_Y$ is independent of t, and since $Cov(X_s, Y_t) = 0$ for all s and t, it follows that $Cov(X_{t+h} + Y_{t+h}, X_t + Y_t) = \gamma_X(h) + \gamma_Y(h)$, which is independent of t.

Exercise 1.10

For $m_t = \sum_{k=0}^{p} c_k t^k$, we have

$$\begin{aligned}\nabla m_t &= \sum_{k=0}^{p} c_k t^k - \sum_{k=0}^{p} c_k (t-1)^k \\ &= p c_p t^{p-1} + \sum_{k=0}^{p-2} b_k t^k\end{aligned}$$

since $(t-1)^p = t^p - pt^{p-1} + \ldots$. The b_k, $k = 0, \ldots, p-2$ are suitable constants. Consequently, ∇m_t is a polynomial of degree $p-1$ and therefore, by successive application of the difference operator ∇, we deduce that $\nabla^{p+1} m_t = 0$

Exercise 1.12

a)

We first prove that a linear filter $\{a_j\}$ passes a polynomial of degree p if and only if $\sum_j a_j = 1$ and $\sum_j (-j)^r a_j = 0$, $r = 1, \ldots, p$. To prove this, it is enough to show that $t^r = \sum_j a_j (t-j)^r$ for $r = 0, \ldots, p$.

$$\sum_j a_j(t-j)^r = \sum_{k=0}^{r} \binom{r}{k} t^k \Big(\sum_j a_j(-j)^{r-k} \Big) = t^r$$

for $r = 0, \ldots, p$ if and only if the above conditions hold.

b)

For Spencer's 15-point moving average filter, $\{a_j, j = -7, \ldots, 7\}$, it is a simple matter to check that

$$\sum_{j=-7}^{7} a_j = 1$$

$$\sum_{j=-7}^{7} (-j)^r a_j = 0, \quad \text{for} \quad r = 1, 2, 3$$

Exercise 1.14

i)

$$a_0 = \frac{3}{9}, \quad a_1 = \frac{4}{9} = a_{-1}, \quad a_2 = -\frac{1}{9} = a_{-2}$$

$$\begin{aligned} \sum a_i &= \frac{3}{9} + \frac{8}{9} - \frac{2}{9} = 1 \\ \sum i a_i &= 0 \\ \sum i^2 a_i &= 0 \\ \sum i^3 a_i &= 0 \end{aligned}$$

By Exercise 1.12(a), the filter passes cubic trend without distortion.

ii)

If $s_t = s_{t-3}$ and $\sum_{t=1}^{3} s_t = 0$, then

$$\begin{aligned} &\frac{3}{9}s_t + \frac{4}{9}s_{t-1} - \frac{1}{9}s_{t-2} + \frac{4}{9}s_{t+1} - \frac{1}{9}s_{t+2} \\ &= \frac{3}{9}(s_t + s_{t+1} + s_{t+2}) = 0 \end{aligned}$$

since $s_{t-2} = s_{t+1}$ and $s_{t-1} = s_{t+2}$. That is, arbitrary seasonal component of period 3 is eliminated.

1.15 (a) Since s_t has period 12,

$$\begin{aligned}\nabla_{12} X_t &= \nabla_{12}(a + bt + s_t + Y_t) \\ &= 12b + Y_t - Y_{t-12}\end{aligned}$$

so that

$$W_t := \nabla \nabla_{12} X_t = Y_t - Y_{t-1} - Y_{t-12} - Y_{t-13}.$$

Then $EW_t = 0$ and

$$\begin{aligned}&\mathrm{Cov}(W_{t+h}, W_t) \\ &\quad = \mathrm{Cov}(Y_{t+h} - Y_{t+h-1} - Y_{t+h-12} + Y_{t+h-13}, Y_t - Y_{t-1} - Y_{t-12} + Y_{t-13}) \\ &\quad = 4\gamma(h) - 2\gamma(h-1) - 2\gamma(h+1) + \gamma(h-11) + \gamma(h+11) - 2\gamma(h-12) \\ &\qquad - 2\gamma(h+12) + \gamma(h+13) + \gamma(h-13)\end{aligned}$$

where $\gamma(\cdot)$ is the autocovariance function of $\{Y_t\}$. Since EW_t and $\mathrm{Cov}(W_{t+h}, W_t)$ are independent of t, $\{W_t\}$ is stationary. Also note that $\{\nabla_{12} X_t\}$ is stationary.

(b) $X_t = (a + bt)s_t + Y_t$

$$\begin{aligned}\nabla_{12} X_t &= bts_t - b(t-12)s_{t-12} + Y_t - Y_{t-12} \\ &= 12bs_{t-12} + Y_t - Y_{t-12}.\end{aligned}$$

Now let $U_t = \nabla_{12}^2 X_t = Y_t - 2Y_{t-12} + Y_{t-24}$. Then $EU_t = 0$ and

$$\begin{aligned}\mathrm{Cov}(U_{t+h}, U_t) &= \mathrm{Cov}(Y_{t+h} - 2Y_{t+h-12} + Y_{t+h-24}, Y_t - 2Y_{t-12} + Y_{t-24}) \\ &= 6\gamma(h) - 4\gamma(h+12) - 4\gamma(h-12) + \gamma(h+24) + \gamma(h-24)\end{aligned}$$

which is independent of t. Hence $\{U_t\}$ is stationary.