TMA4285 Tidsrekker og filterteori

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Exercise 2.1

$S(a,b) = E[(X_{n+h} - aX_n - b)^2]$ to be minimized wrt a and b. Now

$S(a,b) = E[\big((X_{n+h}-\mu) - a(X_n-\mu) - b - a\mu + \mu\big)] = \gamma(0) + a^2\gamma(0) + (b + a\mu - \mu)^2 - 2a\gamma(h)$

This gives

$$\frac{\partial S}{\partial a} = 2a\gamma(0) + 2\mu(b + a\mu - \mu) - 2\gamma(h)$$

$$\frac{\partial S}{\partial b} = 2(b + a\mu - \mu)$$

S is clearly minimized wrt b when for $b = \mu(1-a)$. Substituting this value into $\frac{\partial S}{\partial a}$ and equating to zero leads to the result

$$a = \frac{\gamma(h)}{\gamma(0)} = \rho(h)$$

Hence, $S(a,b)$ is minimized when

$$a = \rho(h), \quad b = \mu(1 - \rho(h))$$

The BLP (best linear predictor) of X_{n+h} in terms of X_n is therefore $\mu + \rho(h)(X_n - \mu)$.

Exercise 2.3

a)

$$X_t = Z_t + 0.3Z_{t-1} - 0.4Z_{t-2}$$

$$\begin{aligned}
\gamma(0) &= 1 + 0.3^2 + 0.4^2 = 1.25 \\
\gamma(1) &= 0.3 - 0.4 \cdot 0.3 = 0.18 \\
\gamma(2) &= -0.4 \\
\gamma(h) &= 0, \quad h > 2 \\
\gamma(-h) &= \gamma(h)
\end{aligned}$$

b)

$$Y_t = \tilde{Z}_t - 1.2\tilde{Z}_{t-1} - 1.6\tilde{Z}_{t-2}$$

$$\begin{aligned}
\gamma(0) &= 0.25(1 + 1.2^2 + 1.6^2) = 1.25 \\
\gamma(1) &= 0.25(-1.2 + 1.6 \cdot 1.2) = 0.18 \\
\gamma(2) &= -1.6 \cdot 0.25 = -0.4 \\
\gamma(h) &= 0, \quad h > 2 \\
\gamma(-h) &= \gamma(h)
\end{aligned}$$

That is, we obtain the same ACVF as in a).

Exercise 2.5

$\sum_{j=1}^{\infty} \theta^j X_{n-j}$ converges absolutely (with probability 1) since

$$\begin{aligned}
E[\sum_{j=1}^{\infty} |\theta|^j |X_{n-j}|] &\leq \sum_{j=1}^{\infty} |\theta|^j E[|X_{n-j}|] \\
&\leq \sum_{j=1}^{\infty} |\theta|^j \sqrt{\gamma(0) + \mu^2} \quad \text{by Cauchy-Schwartz inequality} \\
&< \infty \quad \text{since} |\theta| < 1
\end{aligned}$$

That is, $\sum_{j=1}^{\infty} |\theta|^j |X_{n-j}| < \infty$ with probability 1.

Mean square convergence of $S_m = \sum_{j=1}^m \theta^j X_{n-j}$ as $m \to \infty$ can be verified by invoking Cauchy's criterion. For $m > k$

$$\begin{aligned}
E[|S_m - S_k|^2] &= E[(\sum_{j=k+1}^{m} \theta^j X_{n-j})^2] \\
&= \sum_{i=k+1}^{m} \sum_{j=k+1}^{m} \theta^{i+j} E[X_{n-i} X_{n-j}]
\end{aligned}$$

$$\begin{aligned}
E[|S_m - S_k|^2] = E[(\sum_{j=k+1}^{m} \theta^j X_{n-j})^2] &= \sum_{i=k+1}^{m} \sum_{j=k+1}^{m} \theta^{i+j} E[X_{n-i} X_{n-j}] \\
&= \sum_{i=k+1}^{m} \sum_{j=k+1}^{m} \theta^{i+j} \big(\gamma(i-j) + \mu^2\big) \\
&\leq \sum_{i=k+1}^{m} \sum_{j=k+1}^{m} |\theta|^{i+j} \big(\gamma(0) + \mu^2\big) = \big(\gamma(0) + \mu^2\big) \Big(\sum_{j=k+1}^{m} |\theta|^j\Big)^2 \\
&\to 0 \quad \text{as} \quad k, m \to \infty
\end{aligned}$$

since $\sum_{j=1}^{\infty} |\theta|^j < \infty$. Hence, by Cauchy's mutual convergence criterion, mean square convergence is guaranteed.

Exercise 2.7

$$\begin{aligned}
\frac{1}{1 - \phi z} &= \frac{-\frac{1}{\phi z}}{1 - \frac{1}{\phi z}} \\
&= -\frac{1}{\phi z} \Big(1 + \frac{1}{\phi z} + \frac{1}{(\phi z)^2} + \dots\Big) \\
&= -\sum_{j=1}^{\infty} (\phi z)^{-j}
\end{aligned}$$

since $|\phi z| > 1$.

Exercise 2.8

$$X_t = \phi X_{t-1} + Z_t$$

$$
\begin{aligned}
X_t &= \phi X_{t-1} + Z_t \\
&= Z_t + \phi(Z_{t-1} + \phi X_{t-2}) \\
&= \dots \\
&= Z_t + \phi Z_{t-1} + \dots + \phi^n Z_{t-n} + \phi^{n+1} X_{t-n-1})
\end{aligned}
$$

That is

$$X_t - \phi^{n+1} X_{t-n-1} = Z_t + \phi Z_{t-1} + \dots + \phi^n Z_{t-n}$$

First we calculate

$$
\begin{aligned}
\mathrm{Var}(X_t - \phi^{n+1} X_{t-n-1}) &= \gamma(0)(1 + \phi^{2n+2}) - 2\phi^{n+1}\gamma(n+1) \\
&\leq \gamma(0)(1 + |\phi|^{2n+2} + 2|\phi|^{n+1}) = 4\gamma(0)
\end{aligned}
$$

if X_t is stationary and $|\phi| = 1$

Next we calculate

$$\mathrm{Var}(Z_t + \phi Z_{t-1} + \dots + \phi^n Z_{t-n}) = n\sigma^2$$

if $|\phi| = 1$

Since clearly $n\sigma^2 > 4\gamma(0)$ for sufficiently large n, we have reached a contradiction. Hence X_t cannot be stationary if $|\phi| = 1$.

Exercise 2.10

$$X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1}$$

where $\phi = \theta = 0.5$

According to Section 2.3, equation (2.3.3), we obtain that

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$$

where $\psi_0 = 1$, $\psi_j = (\phi + \theta)\phi^{j-1} = 0.5^{j-1}$ for $j = 1, 2, \dots$.

From Section 2.3, equation (2.3.5), we get

$$Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j}$$

where $\pi_0 = 1$, $\pi_j = -(\phi + \theta)(-\theta)^{j-1} = -(-0.5)^{j-1}$ for $j = 1, 2, \dots$.

Agrees with the results from ITSM.