Exercise 2.12

The given MA(1)-model is

$$X_t = Z_t - 0.6 Z_{t-1}$$

where $Z_t \sim \mathrm{WN}(0,1)$.

Observed that $\overline{x}_{100} = 0.157$

The variance of $\overline{x}_{100}$:

$$\begin{aligned}
\mathrm{Var}[\overline{x}_{100}] &= \frac{1}{n}\sum_{h=-n}^{n}\Big(1 - \frac{|h|}{n}\Big)\gamma(h) \\
&= \frac{1}{100}\Big(\gamma(0) + 2\cdot\frac{99}{100}\gamma(1)\Big) \\
&= \frac{1}{100}\Big(1.36 - 1.98\cdot 0.6\Big) \\
&= 0.00172
\end{aligned}$$

That is, 95% confidence bounds for μ are approximately

$$\begin{aligned}
&\overline{x}_{100} \pm 1.96\sqrt{0.00172} \\
&= 0.157 \pm 1.96\cdot 0.0415 = 0.157 \pm 0.0813 = 0.076,\ 0.238
\end{aligned}$$

Reject H_0: $\mu = 0$ in favour of the alternative hypothesis H_1: $\mu \neq 0$ at significance level 0.05 since the 95% bounds for μ do not include the value 0.

Note: The conclusion would differ if the time series $X_t \sim IID(0, 1.36)$.

Exercise 2.13

a)

Assume an AR(1)-model

$$X_t = \phi X_{t-1} + Z_t$$

Since $\rho(h) = \phi^h$ $(h > 0)$ for an AR(1)-model, and it has been observed that $\hat{\rho(2)} = 0.145$, we shall assume that $\phi^2 << 1$. Using Bartlett's formula, the following approximate relations are obtained:

$$\mathrm{Var}[\hat{\rho(1)}] \approx \frac{1}{n}\big(1 - \phi^2\big)$$

and

$$\mathrm{Var}[\hat{\rho(2)}] \approx \frac{1}{n}\big(1 - \phi^2\big)\big(1 + 3\phi^2\big)$$

That is, 95% confidence bounds for $\rho(1)$ are approximately

$$\hat{\rho(1)} \pm \frac{1.96}{\sqrt{n}}\sqrt{1 - \phi^2}$$

Correspondingly, 95% confidence bounds for $\rho(2)$ are approximately

$$\hat{\rho(2)} \pm \frac{1.96}{\sqrt{n}}\sqrt{(1 - \phi^2)(1 + 3\phi^2)}$$

With $\phi = \hat{\phi} = \hat{\rho(1)}$, $n = 100$, $\hat{\rho(1)} = 0.438$, $\hat{\rho(2)} = 0.145$, these bounds become for $\rho(1)$: 0.262, 0.614, and for $\rho(2)$: -0.073, 0.369.

These values are not consistent with $\phi = 0.8$, since both $\rho(1) = 0.8$ and $\rho(2) = 0.64$ are outside these bounds.

b)

Assume an MA(1)-model

$$X_t = Z_t + \theta Z_{t-1}$$

Bartlett's formula gives the following approximate relations

$$\mathrm{Var}[\hat{\rho(1)}] \approx \frac{1}{n}\big(1 - 3\rho(1)^2 + 4\rho(1)^4\big)$$

and

$$\mathrm{Var}[\hat{\rho(2)}] \approx \frac{1}{n}\big(1 + 2\rho(1)^2\big)$$

That is, 95% confidence bounds for $\rho(1)$ are approximately

$$\hat{\rho(1)} \pm \frac{1.96}{\sqrt{n}}\sqrt{1 - 3\rho(1)^2 + 4\rho(1)^4}$$

Correspondingly, 95% confidence bounds for $\rho(2)$ are approximately

$$\hat{\rho(2)} \pm \frac{1.96}{\sqrt{n}}\sqrt{1+2\rho(1)^2}$$

With the numbers as in a), it is now obtained that these bounds become for $\rho(1)$: 0.290, 0.586, and for $\rho(2)$: -0.082, 0.378.

$\theta = 0.6$ leads to $\rho(1) = \frac{\theta}{1+\theta^2} = 0.4412$, $\rho(2) = 0$. It follows that the confidence bounds are consistent with these two values, and the data are therefore consistent with the MA(1)-model $X_t = Z_t + 0.6Z_{t-1}$

Exercise 2.14

$$X_t = A\cos(\omega t) + B\sin(\omega t), \quad t \in \mathbb{Z}$$

where A and B are uncorrelated random variables with zero mean and variance 1. This process is stationary with ACF $\rho(h) = \cos(\omega h)$.

a)

$$P_1 X_2 = \phi_{11} X_1$$

where $\gamma(0)\phi_{11} = \gamma(1)$, which gives $\phi_{11} = \rho(1) = \cos\omega$. Hence

$$P_1 X_2 = \cos(\omega) X_1$$

Also

$$E[(X_2 - P_1X_2)^2] = \gamma(0) - \phi_{11}\gamma(1) = \gamma(0)(1-\cos^2\omega) = \sin^2\omega$$

Note: 2.14 is an example in which the matrix Γ_n in the equation $\Gamma_n \overline{\phi}_n = \overline{\gamma}_n$ is singular for $n \geq 3$. This is because $X_3 = (2\cos\omega)X_2 - X_1$.

b)

$$P_2 X_3 = \phi_{21} X_2 + \phi_{22} X_1$$

where

$$\gamma(0)\phi_{21} + \gamma(1)\phi_{22} = \gamma(1)$$
$$\gamma(1)\phi_{21} + \gamma(0)\phi_{22} = \gamma(2)$$

that is

$$\phi_{21} + (\cos\omega)\phi_{22} = \cos\omega$$
$$(\cos\omega)\phi_{21} + \phi_{22} = \cos 2\omega$$

Solving these equations give $\phi_{22}(\cos^2\omega - 1) = \cos^2\omega - 2\cos^2\omega + 1 = -\cos^2\omega + 1$, that is, $\phi_{22} = -1$, and then, $\phi_{21} = \cos\omega - \phi_{22}\cos\omega = 2\cos\omega$. Hence

$$P_2 X_3 = (2\cos\omega)X_2 - X_1$$

and

$$\begin{aligned} E[(X_3 - P_2X_3)^2] &= \gamma(0) - \overline{\phi}_2\overline{\gamma}_2 \\ &= 1 - (2\cos\omega, -1)(\cos\omega, \cos 2\omega) \\ &= 1 - 2\cos^2\omega + \cos 2\omega = 0 \end{aligned}$$

c)

From b) and stationarity, it follows that

$$P(X_{n+1}|X_n, X_{n-1}) = (2\cos\omega)X_n - X_{n-1}$$

with MSE = 0.

Since $(2\cos\omega)X_n - X_{n-1}$ is a linear combination of X_s, $-\infty < s \leq n$, and since it is impossible to find a predictor of this form with smaller MSE, we conclude that $\tilde{P}_n X_{n+1} = (2\cos\omega)X_n - X_{n-1}$ with MSE = 0.

Exercise 2.18

Given the MA(1) process

$$X_t = Z_t - \theta Z_{t-1}$$

where $|\theta| < 1$, and $Z_t \sim WN(0, \sigma^2)$. Represented as an AR(∞) process, it assumes the form

$$Z_t = X_t + \theta X_{t-1} + \theta^2 X_{t-2} + \ldots$$

Setting $t = n+1$ in the last equation and applying $\tilde{P}_n$ to each side, leads to the result

$$\tilde{P}_n X_{n+1} = -\sum_{j=1}^{\infty} \theta^j X_{n+1-j} = -\theta Z_n$$

Prediction error = $X_{n+1} - \tilde{P}_n X_{n+1} = Z_{n+1}$. Hence, MSE = $E[Z_{n+1}^2] = \sigma^2$.