Exercise 3.1

a)

$$\phi(z) = 1 + 0.2z - 0.48z^2 = (1 + 0.8z)(1 - 0.6z) = 0$$

when $z = -\frac{1}{0.8} = -1.25$ or $z = \frac{1}{0.6} = 1.67$. Since both of these zeroes are outside the unit circle, the process is causal. The process is obviously invertible since $Z_t = X_t + 0.2X_{t-1} - 0.48X_{t-2}$.

b)

$$\phi(z) = 1 + 1.92z + 0.88z^2 = (1 + 1.1z)(1 + 0.8z) = 0$$

when $z = -\frac{1}{1.1} = -0.91$ or $z = -\frac{1}{0.8} = -1.25$. Since the first of these zeroes lies inside the unit circle, the process is not causal.

$$\theta(z) = 1 + 0.2z + 0.7z^2 = 0$$

when $z = \frac{-0.2 \pm i\sqrt{2.76}}{1.4}$. Then $|z|^2 = \frac{(-0.2)^2+2.76}{1.4^2} = 1.429 > 1$, which implies that the process is invertible.

c)

$$\phi(z) = 1 + 0.6z = 0$$

when $z = -\frac{1}{0.6} = -1.67$. Since the zero lies outside the unit circle, the process is causal.

$$\theta(z) = 1 + 1.2z = 0$$

when $z = -\frac{1}{1.2} = -0.83$. Since the zero lies inside the unit circle, the process is not invertible.

d)

$$\phi(z) = (1 + 0.9z)^2 = 0$$

when $z = -\frac{1}{0.9} = -1.11$. There is only one zero (more precisely, two coalescing zeros), which lies outside the unit circle. Hence the process is causal. AR(p)-processes are always invertible.

e)

$$\phi(z) = 1 + 1.6z = 0$$

when $z = -\frac{1}{1.6} = -0.625$. The zero lies inside the unit circle, hence the process is not causal.

$$\theta(z) = 1 - 0.4z + 0.04z^2 = (1 - 0.2z)^2 = 0$$

when $z = \frac{1}{0.2} = 5$. Since the only zero lies outside the unit circle, the process is invertible.

Exercise 3.2 (a)

===============

ITSM::(ACF/PACF)

===============

\# of Lags = 40

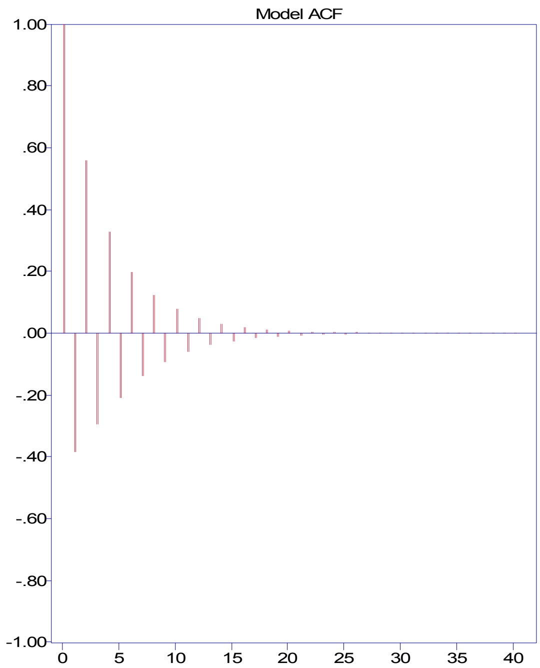
Model Autocorrelations:

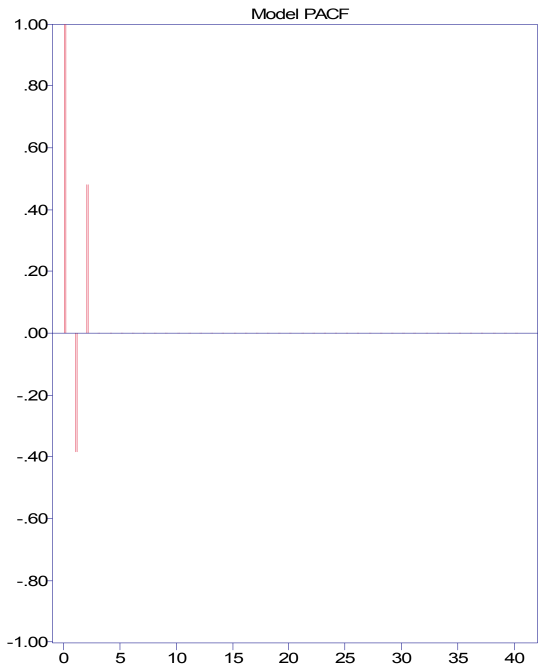
Gamma(0) = 1.52496246

1.0000	-.3846	.5569	-.2960	.3265
-.2074	.1982	-.1392	.1230	-.0914
.0773	-.0593	.0490	-.0383	.0312
-.0246	.0199	-.0158	.0127	-.0101
.0081	-.0065	.0052	-.0041	.0033
-.0027	.0021	-.0017	.0014	-.0011
.0009	-.0007	.0006	-.0004	.0004
-.0003	.0002	-.0002	.0001	-.0001

Model Partial Autocorrelations:

1.0000	-.3846	.4800	.2116E-16	-.1058E-16
-.6349E-16	-.3174E-16	.1058E-16	.5291E-17	-.2645E-16
.1058E-16	.4233E-16	-.1587E-16	-.3704E-16	-.1323E-17
.1190E-16	.2645E-17	-.1323E-17	.0000	-.3968E-17
.3968E-17	.6283E-17	-.2976E-17	-.4299E-17	-.3307E-18
.1819E-17	.9920E-18	.8267E-19	-.8267E-18	.0000
.4547E-18	-.8267E-19	.2893E-18	.1447E-18	-.4340E-18
-.2067E-18	.9300E-19	.3100E-19	.1550E-18	.5167E-19





Exercise 3.2 (c)

===============

ITSM::(ACF/PACF)

===============

\# of Lags = 40

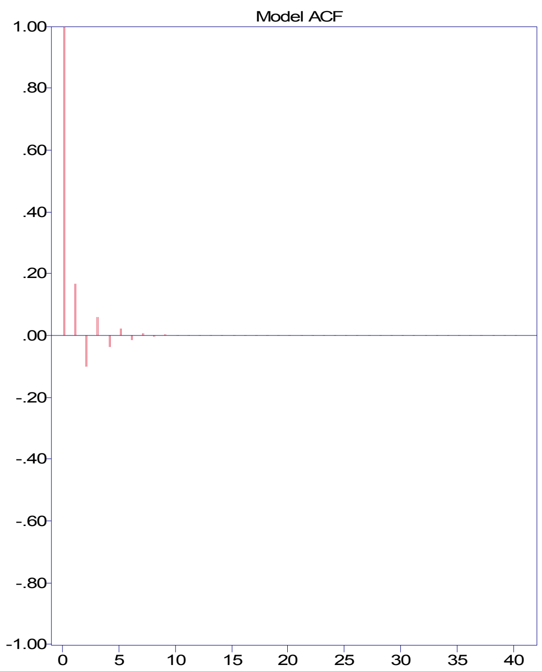
Model Autocorrelations:

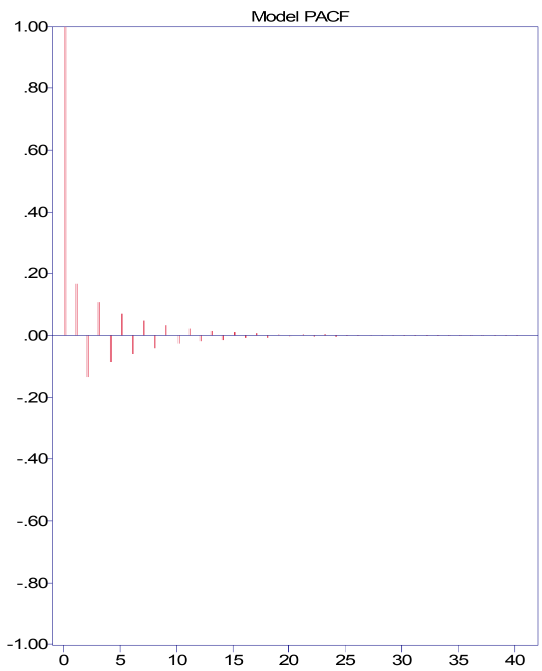
Gamma(0) = 1.56250000

1.0000	.1680	-.1008	.0605	-.0363
.0218	-.0131	.0078	-.0047	.0028
-.0017	.0010	-.0006	.0004	-.0002
.0001	-.7899E-04	.4739E-04	-.2844E-04	.1706E-04
-.1024E-04	.6142E-05	-.3685E-05	.2211E-05	-.1327E-05
.7960E-06	-.4776E-06	.2866E-06	-.1719E-06	.1032E-06
-.6190E-07	.3714E-07	-.2228E-07	.1337E-07	-.8022E-08
.4813E-08	-.2888E-08	.1733E-08	-.1040E-08	.6238E-09

Model Partial Autocorrelations:

1.0000	.1680	-.1328	.1068	-.0869
.0713	-.0587	.0486	-.0403	.0334
-.0278	.0231	-.0192	.0160	-.0133
.0111	-.0093	.0077	-.0064	.0054
-.0045	.0037	-.0031	.0026	-.0022
.0018	-.0015	.0012	-.0010	.0009
-.0007	.0006	-.0005	.0004	-.0003
.0003	-.0002	.0002	-.0002	.0001





Exercise 3.2 (d)

===============

ITSM::(ACF/PACF)

===============

# of Lags = 40

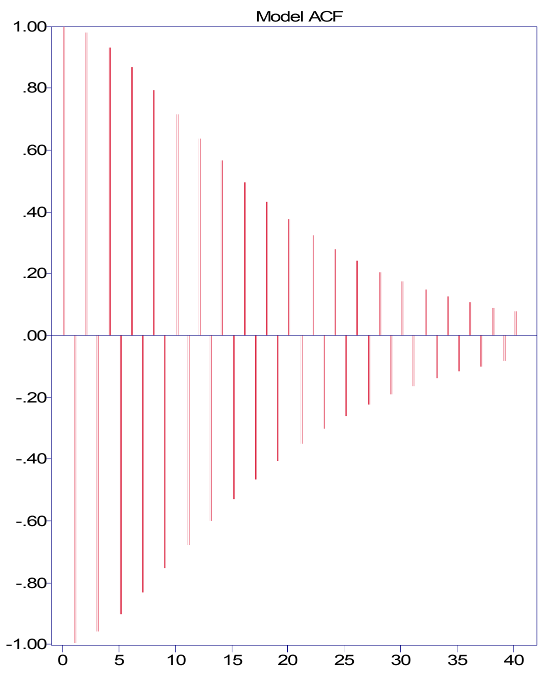
Model Autocorrelations:

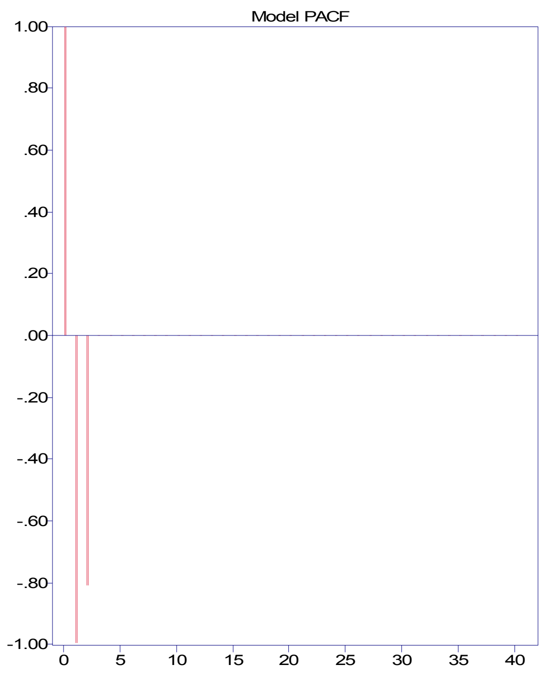
Gamma(0) = .26388686E+03

1.0000	-.9945	.9801	-.9586	.9316
-.9004	.8662	-.8298	.7920	-.7534
.7147	-.6762	.6382	-.6011	.5650
-.5301	.4965	-.4644	.4337	-.4045
.3768	-.3506	.3259	-.3026	.2807
-.2602	.2410	-.2230	.2062	-.1905
.1759	-.1623	.1497	-.1380	.1271
-.1170	.1077	-.0990	.0910	-.0837

Model Partial Autocorrelations:

1.0000	-.9945	-.8100	.0000	.5859E-13
.1172E-12	.0000	-.1172E-12	-.5859E-13	.0000
.0000	.0000	.0000	.0000	-.5859E-13
-.1172E-12	-.1172E-12	-.1172E-12	-.8789E-13	-.2930E-13
.0000	.0000	-.2930E-13	-.8789E-13	-.1465E-12
-.1172E-12	-.1465E-13	.2930E-13	.4395E-13	.2930E-13
-.2930E-13	-.2930E-13	-.1465E-13	-.1465E-13	.1465E-13
.2930E-13	.1465E-13	.1465E-13	.1465E-13	-.7324E-14





Exercise 3.3 (a)

==========================
ITSM::(AR/MA Infinity)
==========================

ARMA Model:

ARMA Model:
X(t) = - .2000 X(t-1) + .4800 X(t-2)
+ Z(t)

AR/MA Infinity:

	MA-Infinity	AR-Infinity
0	1.00000	1.00000
1	-.20000	.20000
2	.52000	-.48000
3	-.20000	.00000
4	.28960	.00000
5	-.15392	.00000
6	.16979	.00000
7	-.10784	.00000
8	.10307	.00000
9	-.07238	.00000
10	.06395	.00000
11	-.04753	.00000
12	.04020	.00000
13	-.03085	.00000
14	.02547	.00000
15	-.01990	.00000
16	.01621	.00000
17	-.01279	.00000
18	.01034	.00000
19	-.00821	.00000
20	.00660	.00000
21	-.00526	.00000
22	.00422	.00000
23	-.00337	.00000
24	.00270	.00000
25	-.00216	.00000
26	.00173	.00000
27	-.00138	.00000
28	.00111	.00000
29	-.00088	.00000
30	.00071	.00000
31	-.00057	.00000
32	.00045	.00000
33	-.00036	.00000
34	.00029	.00000
35	-.00023	.00000
36	.00019	.00000

37	-.00015	.00000
38	.00012	.00000
39	-.00009	.00000
40	.00008	.00000
41	-.00006	.00000
42	.00005	.00000
43	-.00004	.00000
44	.00003	.00000
45	-.00002	.00000
46	.00002	.00000
47	-.00002	.00000
48	.00001	.00000
49	-.00001	.00000
50	.61168E-05	.00000

Exercise 3.3 (c)

=========================

ITSM::(AR/MA Infinity)

=========================

ARMA Model:

ARMA Model:
X(t) = - .6000 X(t-1)
+ Z(t) + 1.200 Z(t-1)

AR/MA Infinity:

	MA-Infinity	Non-Invertible
0	1.00000	*
1	.60000	*
2	-.36000	*
3	.21600	*
4	-.12960	*
5	.07776	*
6	-.04666	*
7	.02799	*
8	-.01680	*
9	.01008	*
10	-.00605	*
11	.00363	*
12	-.00218	*
13	.00131	*
14	-.00078	*
15	.00047	*
16	-.00028	*
17	.00017	*
18	-.00010	*
19	.00006	*

```
20       -.00004       *
21        .00002       *
22       -.00001       *
23     .78973E-05       *
24    -.47384E-05       *
25     .28430E-05       *
26    -.17058E-05       *
27     .10235E-05       *
28    -.61409E-06       *
29     .36846E-06       *
30    -.22107E-06       *
31     .13264E-06       *
32    -.79587E-07       *
33     .47752E-07       *
34    -.28651E-07       *
35     .17191E-07       *
36    -.10314E-07       *
37     .61887E-08       *
38    -.37132E-08       *
39     .22279E-08       *
40    -.13367E-08       *
41     .80205E-09       *
42    -.48123E-09       *
43     .28874E-09       *
44    -.17324E-09       *
45     .10395E-09       *
46    -.62367E-10       *
47     .37420E-10       *
48    -.22452E-10       *
49     .13471E-10       *
50    -.80828E-11       *
```

Exercise 3.3 (d)

```
==========================
ITSM::(AR/MA Infinity)
==========================

ARMA Model:

ARMA Model:
X(t) = - 1.800 X(t-1) - .8100 X(t-2)
     + Z(t)

AR/MA Infinity:
          MA-Infinity    AR-Infinity
    0       1.00000        1.00000
    1      -1.80000        1.80000
    2       2.43000         .81000
```

3	-2.91600	.00000
4	3.28050	.00000
5	-3.54294	.00000
6	3.72009	.00000
7	-3.82638	.00000
8	3.87420	.00000
9	-3.87420	.00000
10	3.83546	.00000
11	-3.76573	.00000
12	3.67158	.00000
13	-3.55861	.00000
14	3.43152	.00000
15	-3.29426	.00000
16	3.15013	.00000
17	-3.00189	.00000
18	2.85180	.00000
19	-2.70170	.00000
20	2.55311	.00000
21	-2.40722	.00000
22	2.26497	.00000
23	-2.12711	.00000
24	1.99416	.00000
25	-1.86653	.00000
26	1.74449	.00000
27	-1.62819	.00000
28	1.51771	.00000
29	-1.41304	.00000
30	1.31413	.00000
31	-1.22087	.00000
32	1.13312	.00000
33	-1.05071	.00000
34	.97345	.00000
35	-.90114	.00000
36	.83355	.00000
37	-.77047	.00000
38	.71167	.00000
39	-.65693	.00000
40	.60602	.00000
41	-.55872	.00000
42	.51482	.00000
43	-.47411	.00000
44	.43640	.00000
45	-.40149	.00000
46	.36919	.00000
47	-.33934	.00000
48	.31177	.00000
49	-.28632	.00000
50	-.25253	.00000

Exercise 3.5

$$Y_t = X_t + W_t$$

where $W_t \sim \text{WN}(0, \sigma_w^2)$, X_t is the ARMA(p,q) process satisfying

$$\phi(B)X_t = \theta(B)Z_t\,; \quad Z_t \sim \text{WN}(0, \sigma_z^2)$$

and $E(W_s Z_t) = 0$ for all $s, t \in \mathbb{Z}$

a) Since X_t is an ARMA(p,q) process, we know that it is stationary and that, in fact, $X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}$, where $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$. This immediately implies that $E[X_t] = 0$, and therefore also $E[Y_t] = 0$.

$$\begin{aligned} Cov(Y_{t+h}, Y_t) &= E[Y_{t+h}Y_t] \\ &= E[X_{t+h}X_t] + E[X_{t+h}W_t] + E[W_{t+h}X_t] + E[W_{t+h}W_t] \\ &= \gamma_X(h) + \gamma_W(h) \end{aligned}$$

This result is obtained because $E(W_s X_t) = 0$ for all $s, t \in \mathbb{Z}$, which follows from the fact that $E(W_s X_t) = E(\sum_{j=-\infty}^{\infty} \psi_j W_s Z_{t-j}) = \sum_{j=-\infty}^{\infty} \psi_j E(W_s Z_{t-j}) = 0$ (the next to the last equality holds true since $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$).

We conclude that the covariance function of Y_t only depends on h. Hence, Y_t is a stationary process with ACVF

$$\gamma_Y(h) = \gamma_X(h) + \sigma_w^2 \delta(h)$$

where $\delta(h) = 1$ for $h = 0$ and $\delta(h) = 0$ for $h \neq 0$.

b)

$$U_t = \phi(B)Y_t = \phi(B)X_t + \phi(B)W_t = \theta(B)Z_t + \phi(B)W_t$$

The two processes $\tilde{Z}_t = \theta(B)Z_t$ and $\tilde{W} = \phi(B)W_t$ are uncorrelated, that is, $E[\tilde{Z}_s \tilde{W}_t] = 0$ for all $s, t \in \mathbb{Z}$. $\tilde{Z}_t$ is an MA(q)-process and $\tilde{W}_t$ is an MA(p) process. It follows that U_t is a stationary process with ACVF

$$\gamma_U(h) = \gamma_{\tilde{Z}}(h) + \gamma_{\tilde{W}}(h)$$

We know that $\gamma_{\tilde{Z}}(h) = 0$ for $h > q$, and $\gamma_{\tilde{W}}(h) = 0$ for $h > p$, hence $\gamma_U(h) = 0$ for $h > \max(p, q)$. We conclude that there exists an $r \leq \max(p, q)$ such that $\gamma_U(r) \neq 0$, while $\gamma_U(h) = 0$ for $h > r$. By Proposition 2.1.1, U_t is an MA(r) process, that is, $U_t = \breve{\theta}(B)\breve{Z}_t$

for some MA polynomial $\check{\theta}(z) = 1 + \check{\theta}_1 z + \ldots + \check{\theta}_r z^r$, and $\check{Z}_t \sim \mathrm{WN}(0, \sigma^2)$, for suitable σ^2. Rewriting this in terms of Y_t, we get

$$\phi(B)Y_t = \check{\theta}(B)\check{Z}_t$$

Since Y_t is stationary, we conclude that Y_t in fact is an ARMA(p,r) process

Exercise 3.9

a) The ACVF $\gamma(\cdot)$ of Y_t is given by

$$\gamma(h) = \begin{cases} \sigma^2(1+\theta_1^2+\theta_{12}^2) & : \quad h = 0 \\ \sigma^2\theta_1 & : \quad h = \pm 1 \\ \sigma^2\theta_1\theta_{12} & : \quad h = \pm 11 \\ \sigma^2\theta_{12} & : \quad h = \pm 12 \\ 0 & : \quad \text{otherwise} \end{cases}$$

b) ITSM is used to compute the sample mean and the first 20 values of the empirical ACVF

for the time series $\nabla\nabla_{12}X_t$ where X_t is represented by the DEATHS.TSM data file. This result in the following values. The sample mean is found to be 28.8305, and the sample variance $\hat{\gamma}(0) = 152669$. The empirical ACF $\hat{\rho}(h)$ is then

h	0	1	2	3	4	5	6
$\hat{\rho}(h)$	1.0000	-0.3558	- 0.0987	0.0955	- 0.1125	0.0415	0.1141

h	7	8	9	10	11	12	13
$\hat{\rho}(h)$	- 0.2041	-0.0071	0.1001	- 0.0814	0.1952	- 0.3332	0.0902

h	14	15	16	17	18	19	20
$\hat{\rho}(h)$	0.1163	- 0.0406	- 0.0633	0.1833	- 0.1929	0.0242	0.0902

c)

From the tables, the following values are extracted

$$\hat{\gamma}(1) = -152669 \cdot 0.3558 \quad \hat{\gamma}(11) = 152669 \cdot 0.1952 \quad \hat{\gamma}(12) = -152669 \cdot 0.3332$$

Matching $\gamma(1)$, $\gamma(11)$, $\gamma(12)$ with the corresponding empirical values, we find

$$\hat{\theta}_1 = \frac{\hat{\gamma}(11)}{\hat{\gamma}(12)} = \frac{\hat{\rho}(11)}{\hat{\rho}(12)} = -\frac{0.1952}{0.3332} = -0.586$$

$$\hat{\theta}_{12} = \frac{\hat{\gamma}(11)}{\hat{\gamma}(1)} = \frac{\hat{\rho}(11)}{\hat{\rho}(1)} = -\frac{0.1952}{0.3558} = -0.549$$

$$\hat{\sigma}^2 = \frac{\hat{\gamma}(1)}{\hat{\theta}_1} = \hat{\gamma}(0)\frac{\hat{\rho}(1)}{\hat{\theta}_1} = 152669 \cdot \frac{0.3558}{0.586} = 92696$$

The model for $Y_t = \nabla\nabla_{12}X_t$ is then

$$Y_t = 28.8305 + Z_t - 0.586Z_{t-1} - 0.549Z_{t-12}$$

where $Z_t \sim \mathrm{WN}(0, 92696)$

Exercise 3.10

i) From ITSM we find

$$\hat{\gamma}(0) = 676789 = \frac{\hat{\sigma}^2}{1-\hat{\phi}^2}$$

$$\hat{\gamma}(1) = 676789 \cdot 0.7323 = 495613 = \frac{\hat{\sigma}^2\hat{\phi}}{1-\hat{\phi}^2}$$

Solving for $\hat{\sigma}^2$ and $\hat{\phi}$, we obtain $\hat{\sigma}^2 = 313852$ and $\hat{\phi} = 0.7323$. ITSM gives the sample mean as $\hat{\mu} = 4503$, which leads to the model

$$Y_t = 0.7323Y_{t-1} + Z_t$$

where $Z_t \sim \mathrm{WN}(0, 313852)$ and $Y_t = X_t - 4503$

ii) The BLP (best linear predictor) of Y_{31} is

$$\hat{Y}_{31} = \hat{\phi}Y_{30} = 0.7323\,(3885 - 4503) = -452$$

where the last observation (1980) is $X_{30} = 3885$. Hence, $\hat{X}_{31} = \hat{Y}_{31} + \hat{\mu} = 4051$. The mean square prediction error is $E[(Y_{31} - \hat{Y}_{31})^2] = E[Z_{31}^2] = \sigma^2$, which we estimate by $\hat{\sigma}^2 = 313852$.

iii) 95% prediction error bounds are estimated as

$$4051 \pm 1.96\sqrt{313852} = 4051 \pm 1098 = 2953,\ 5149$$