TMA4285 Tidsrekker og filterteori

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Exercise 3.11

The PACF at lag 2, ϕ_{22}, of the given MA(1) process is determined from the BLP P_2X_3 of X_3 in terms of X_2 and X_1 by the relation

$$P_2X_3 = \phi_{21}X_2 + \phi_{22}X_1$$

where $Z_t \sim \mathrm{WN}(0, \sigma^2)$.

The value of ϕ_{22} was calculated in Exercise 2.21(a). It was found that

$$\phi_{22} = a_2 = \frac{-\theta^2}{(1+\theta^2)^2 - \theta^2} = \frac{-\theta^2}{1+\theta^2+\theta^4}$$

Exercise 4.1

$$\begin{aligned}
\int_{-\pi}^{\pi} e^{i(k-h)\lambda} &= \int_{-\pi}^{\pi} \left[\cos(k-h)\lambda + i\sin(k-h)\lambda\right] d\lambda \\
&= \begin{cases} \left[\frac{\sin(k-h)\lambda - i\cos(k-h)\lambda}{k-h}\right]_{-\pi}^{\pi} & = \quad 0 \quad \text{for } k \neq h \\ \int_{-\pi}^{\pi} 1 \, d\lambda & = \quad 2\pi \quad \text{for } k = h \end{cases}
\end{aligned}$$

Exercise 4.4

Since

$$\begin{aligned}
\frac{1}{2\pi}\sum_{-\infty}^{\infty} e^{i\omega h}\gamma(h) &= \left[1 - 0.5(e^{-2i\omega} + e^{2i\omega}) - 0.25(e^{-3i\omega} + e^{3i\omega})\right] \\
&= \frac{1}{2\pi}\left[1 - \cos 2\omega - 0.5\cos 3\omega\right] = -\frac{1}{4\pi} < 0 \quad \text{at } \omega = 0,
\end{aligned}$$

$\gamma(h)$ cannot be an autocorrelation function.

Exercise 4.5

For $Z_t = X_t + Y_t$, with X_t and Y_t uncorrelated and stationary, it follows that $E(Z_t) = E(X_t) + E(Y_t) = \text{constant}$, $Cov(Z_{t+h}, Z_t) = \gamma_X(h) + \gamma_Y(h)$. Hence, Z_t is also stationary, and we can write

$$\gamma_Z(h) = \gamma_X(h) + \gamma_Y(h) = \int_{-\pi}^{\pi} e^{i\omega h} dF_X(\omega) + \int_{-\pi}^{\pi} e^{i\omega h} dF_Y(\omega)$$
$$= \int_{-\pi}^{\pi} e^{i\omega h} d[F_X(\omega) + F_Y(\omega)]$$

Therefore, Z_t has the spectral distribution function $F_Z(\omega) = F_X(\omega) + F_Y(\omega)$.

Exercise 4.6

Let $U_t = A\cos(\pi t/3) + B\sin(\pi t/3)$. Then U_t is a stationary process with mean value zero and covariance function $\gamma_U(h) = \nu^2 \cos(\pi h/3) = \frac{\nu^2}{2}(e^{-i\pi h/3} + e^{i\pi h/3})$. U_t is clearly uncorrelated with Y_t, which is also a stationary process with mean value zero and covariance function $\gamma_Y(h)$, which is given as

$$\gamma_Y(h) = \begin{cases} 7.25\sigma^2 & : \quad h = 0 \\ 2.5\sigma^2 & : \quad h = \pm 1 \\ 0 & : \quad |h| > 1 \end{cases}$$

Invoking Exerecise 4.5, we know that the covariance function of X_t is given as $\gamma_X(h) = \gamma_U(h) + \gamma_Y(h)$. The corresponding spectral distribution function $F_X(\omega) = F_U(\omega) + F_Y(\omega)$, where

$$F_U(\omega) = \begin{cases} 0 & : \quad \omega < -\pi/3 \\ \nu^2/2 & : \quad -\pi/3 \le \omega < \pi/3 \\ \nu^2 & : \quad \pi/3 \le \omega \end{cases}$$

and

$$F_Y(\omega) = \int_{-\pi}^{\omega} f_Y(\lambda) d\lambda = \int_{-\pi}^{\omega} \frac{1}{2\pi} \sum_{h=-1}^{1} \gamma_Y(h) e^{-ih\lambda}\, d\lambda$$
$$= \int_{-\pi}^{\omega} \frac{\sigma^2}{2\pi} \left(7.25 + 5\cos\lambda\right) d\lambda = \frac{\sigma^2}{2\pi} \left(7.25(\omega + \pi) + 5\sin\omega\right)$$

Exercise 4.8

The spectral density of X_t is

$$f_X(\omega) = \frac{1}{2\pi}|1 - 0.99e^{-i3\omega}|^{-2} = \frac{1}{2\pi(1.9801 - 1.98\cos 3\omega)}$$

This spectral density has sharp peaks at the frequencies $\omega = 0, \pm 2\pi/3$, which suggests sample paths that are quite smooth and nearly periodic with period 3. The spectral density of the filtered process $Y_t = \frac{1}{3}(X_{t-1} + X_t + X_{t+1})$ is

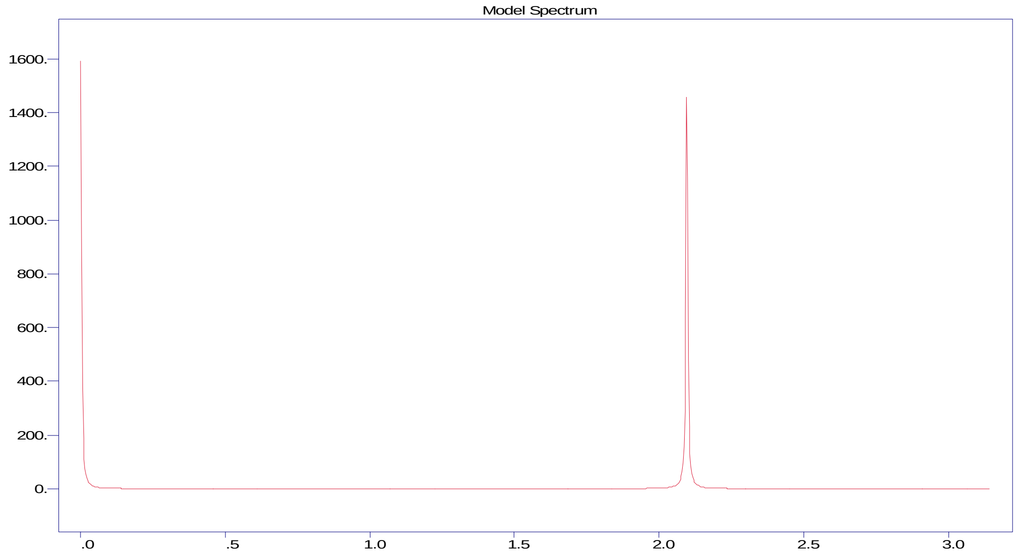
$$f_Y(\omega) = \frac{1}{9}|e^{-i\omega} + 1 + e^{i\omega}|^2 f_X(\omega) = \frac{1}{9}\big(3 + 4\cos\omega + 2\cos 2\omega\big) f_X(\omega)$$

and $f_X(2\pi/3) = 10000/(2\pi)$, $f_Y(2\pi/3) = 0$. So this filter effectively eliminates the strong periodic component of the time series X_t.

The numerical part of this exercise is discussed on the next two pages.

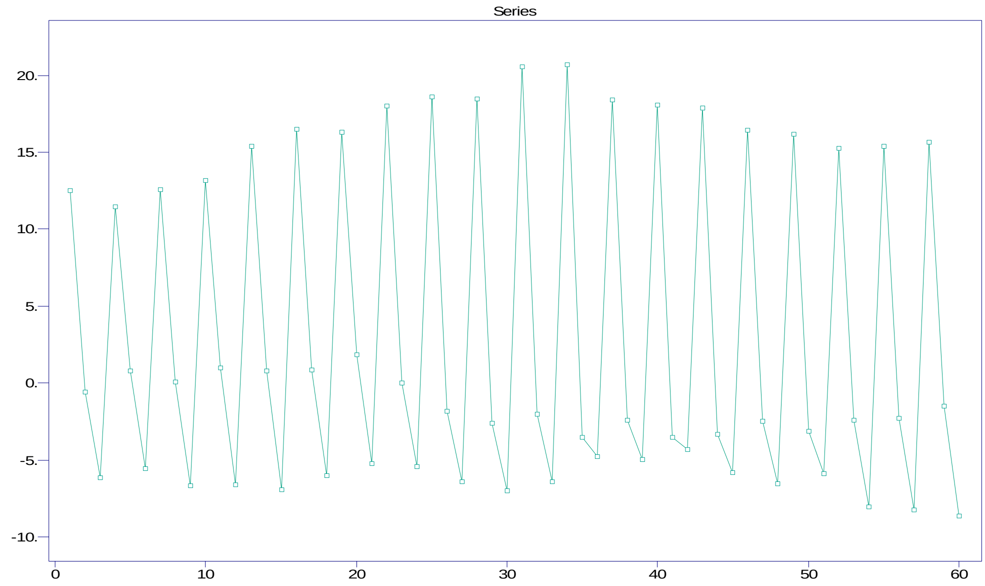
Exercise 4.8

(a) By specifying the given AR(3), ITSM calculates the following spectral density (shown only for positive frequencies)



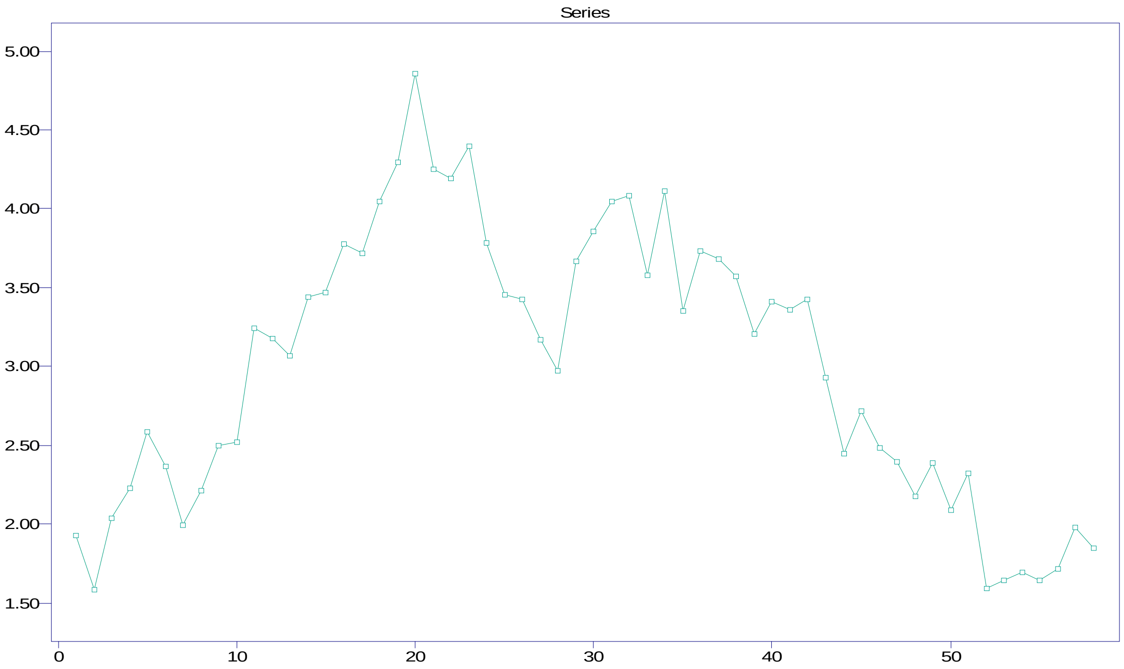
(b) It is seen that the spectral density has a very sharp peak (at non-zero frequencies) at $\omega = 2\pi/3$ and is zero elsewhere (except at zero). This is a strong indication that the time series will exhibit approximately oscillatory behaviour at period 3.

(c) Simulated time series



It is seen that the time series has a period very close to 3.

(e)



A plot of the smoothed time series $Y_t = 1/3\ (X_{t-1} + X_t + X_{t+1})$. The estimates spectral density is shown below. It is clearly seen that the spectral peak at $\omega = 2\pi/3$ has been removed.