**Norges teknisk–**
**naturvitenskapelige universitet**
**Institutt for matematiske fag**



English

Contact during exam:
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# EXAM IN SUBJECT SIF5079 TIME SERIES AND FILTER THEORY

Saturday December 30th 2002
Time: 09:00–14:00

Aids: Calculator HP30S
Statistiske tabeller og formler, Tapir forlag.
K. Rottman: Matematisk formelsamling.
One yellow paper (A5 with stamp) with own formulas and notes.

Grading: December 21st 2002.

**Note**: You should give reasons for all your answers !

**Note**: The Kalman filter equations and some summation formulas are given on the last page.

## Problem 1

Let $\{x_t\}$ follow an ARMA(1,1) model, i.e.

$$(1 - \phi B)x_t = (1 + \theta B)\omega_t, \tag{1}$$

where $B$ denotes the *backshift* operator, i.e. $B^k x_t \equiv x_{t-k}$, and $\{\omega_t\}$ is a white noise process with $\text{Var}(\omega_t) = \sigma_\omega^2$.

**a)** What does it mean that $\{x_t\}$ is invertible?
For what values of $\phi$ and $\theta$ are $\{x_t\}$ invertible?

What does it mean that $\{x_t\}$ is causal?
For what values of $\phi$ and $\theta$ are $\{x_t\}$ causal?

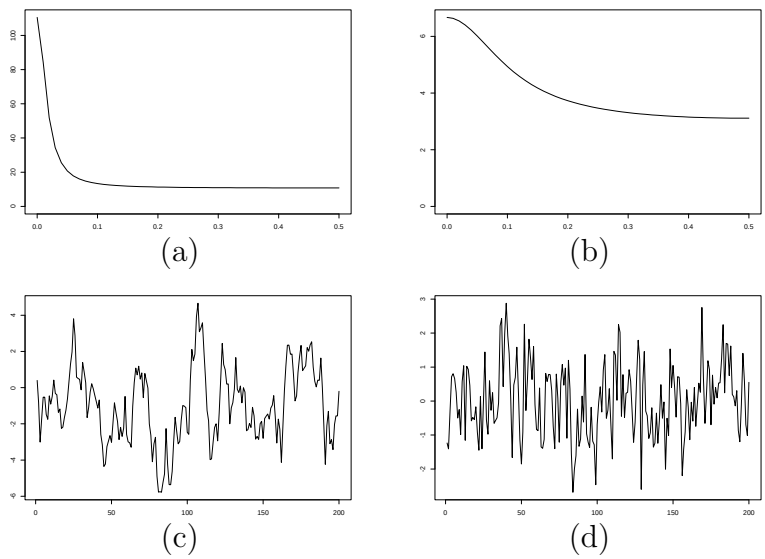


Figure 1: (a) $f(\nu)$ for an AR(1) model with $\sigma_w^2 = 1$ and $\phi = 0.9$, (b) $f(\nu)$ for an AR(1) model with $\sigma_w^2 = 1$ and $\phi = 0.5$, (c) A realisation from an AR(1) model with $\sigma_w^2 = 1$ and $\phi = 0.9$, (d) A realisation from an AR(1) model with $\sigma_w^2 = 1$ and $\phi = 0.5$.

For the rest of this problem, we assume $\{x_t\}$ to be invertible and causal. As well known, this implies in particular that $x_t$ can be expressed as

$$x_t = \sum_{j=0}^{\infty} \psi_j \omega_{t-j}. \tag{2}$$

**b)** Show that the coefficients $\{\psi_j\}$ in equation (2) are given by

$$\psi_0 = 1 \quad \text{and} \quad \psi_j = \phi^{j-1}(\theta + \phi) \quad \text{for } j = 1, 2, 3, \ldots. \tag{3}$$

**c)** Find the autocovariance function, $\gamma(h)$, for $x_t$.

In particular, show that for $\theta = 0$ we get

$$\gamma(h) = \sigma_w^2 \left(1 + \frac{\phi^2}{1 - \phi^2}\right) \phi^{|h|}. \tag{4}$$

In the remainder of this problem we assume $\theta = 0$, so that $x_t$ is a pure AR(1) process. Thus, the autocovariance function is given in equation (4).

**d)** Find the spectral density, $f(\nu)$, for $x_t$.

Figure 1 shows $f(\nu)$ for $\phi = 0.9$ and $\phi = 0.5$. Realisations from these two models are also shown. Using the $f(\nu)$'s, explain the qualitative difference between the two realisations.

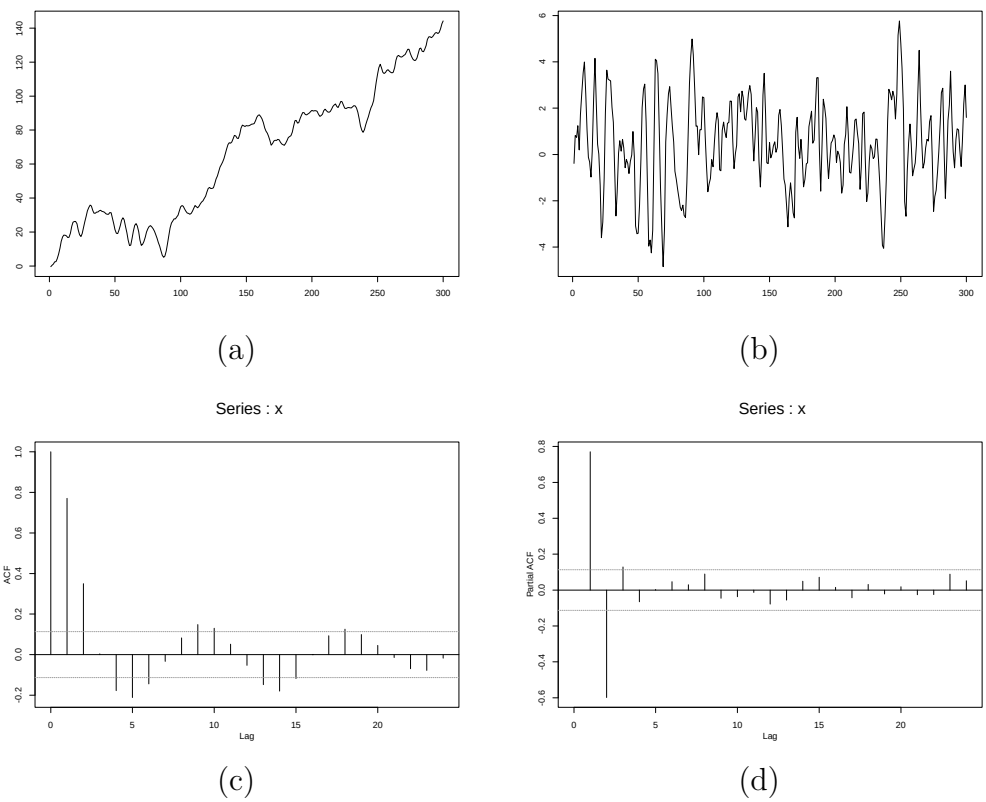


Figure 2: (a) Observed time series, $z_t$. (b) Differenced time series, $x_t = \nabla z_t$. (c) Estimated autocorrelation function for $x_t$. (d) Estimated partial autocorrelation function for $x_t$.

## Problem 2

An observed time series, $z_t; t = 1, 2, \ldots, 300$ is shown in Figure 2(a). The corresponding differenced time series, $x_t = \nabla z_t$ is shown in Figure 2(b). Figures 2(c) and (d) give the estimated autocorrelation function and the estimated partial autocorrelation function for $x_t$, respectively.

**a)** Why is it not reasonable to model the observed time series, $z_t$, as a ARMA($p$,$q$) model? Do you find it natural use an ARMA($p$,$q$) model for the differenced time series, $x_t$? (Remember: Give reasons for your answers.)

Propose values for $p$ and $q$ for an ARMA($p$,$q$) model for $x_t$. (Remember: Give reasons for your answer.)

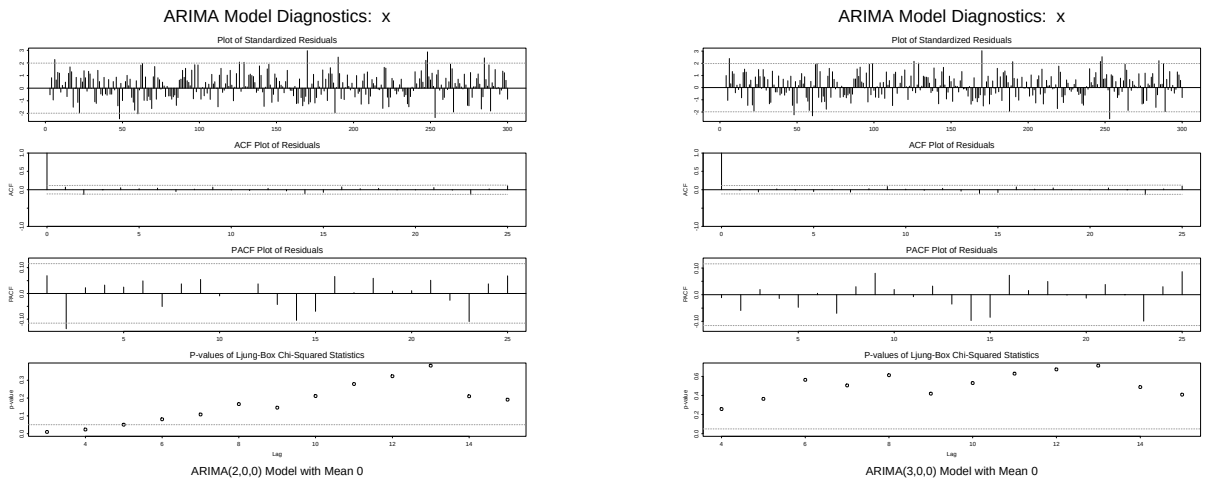


Figure 3: Plots of model diagnostics for the AR(2) model and for the AR(3) model.

Assume we chose to fit an AR(2) model for the time series $x_t$. To estimate the parameters we want to use the method of moments. From the time series $x_t$ we get $\widehat{\gamma}(0) = 3.566$, $\widehat{\gamma}(1) = 2.747$ and $\widehat{\gamma}(2) = 1.248$. It can be assumed known that $\mathrm{E}(x_t) = 0$.

**b)** Explain briefly the method of moments.

For an AR(2) model, deduce a set of homogeneous difference equations for $\gamma(h)$.

For the time series $x_t$, use the method of moments to find estimates for $\sigma_w^2$, $\phi_1$ and $\phi_2$. (It is sufficient that you find a system of equations for the estimates, i.e. you do not have to solve this system of equations.)

In addition to the AR(2) model, we also fit an AR(3) model for $x_t$. Plots of model diagnostics (as produced by Splus) for the two models are given in Figure 3. The AIC values are 835.57 and 827.89 for the AR(2) model and the AR(3) model, respectively.

**c)** Comment on the plots of model diagnostics and on the AIC values. Which of the two proposed models would you prefer?

**Problem 3**

Assume the following scalar state space model

$$x_t = \phi x_{t-1} + \omega_t, \tag{5}$$
$$y_t = x_t + v_t, \tag{6}$$

where $|\phi| < 1$ and where $\{\omega_t\}$ and $\{v_t\}$ are two independent Gaussian white noise processes with $\mathrm{Var}(\omega_t) = 1$ and $\mathrm{Var}(v_t) = \sigma^2$, respectively.

**a)** Use the Kalman filter equations to find

$$P \equiv \lim_{t\to\infty} P_t^t \quad \text{and} \quad K \equiv \lim_{t\to\infty} K_t. \tag{7}$$

Give the answers as functions of $\phi$ and $\sigma^2$.

In particular, show that with $\phi = 0.9$ and $\sigma^2 = 5^2$ one gets $P = 3.043$ and $K = 0.1217$.

Assume $t$ to be large so that your solution in **a**) is valid. Then the Kalman filter is a linear filter and the relations between $\{y_t\}$ and $\{x_t^t\}$ can be expressed as

$$x_t^t = \sum_{j=-\infty}^{\infty} a_j y_{t-j}. \tag{8}$$

**b)** Find an expression for the coefficients $\{a_j\}$. Give the answer as a function of (one or several of) $\phi$, $\sigma^2$, $K$ and $P$.

In particular, show that with $\phi = 0.9$ and $\sigma^2 = 5^2$ one gets $a_j = 0.1217 \cdot 0.7904^j$ for $j = 0, 1, 2, \ldots$.

**c)** Find $|A(\nu)|^2$ for (the steady-state solution of) the Kalman filter for this model. Give the answer as a function of (one or several of) $\phi$, $\sigma^2$, $K$ and $P$.

Sketch $|A(\nu)|^2$ for the following two parameter sets:

1. $\phi = 0.9$ and $\sigma^2 = 5^2$ [then $K = 0.1217$ and $P = 3.043$]
2. $\phi = 0.5$ and $\sigma^2 = 5^2$ [then $K = 0.0498$ and $P = 1.246$]

Comment briefly on the difference between $|A(\nu)|^2$ for the two parameter sets. In particular, use the plots in Figure 1 to explain the qualitative difference between $|A(\nu)|^2$ for the two parameter sets.

Appendix:

The Kalman filter equations:

$$\begin{aligned} x_t^{t-1} &= \phi x_{t-1}^{t-1} \\ P_t^{t-1} &= \phi P_{t-1}^{t-1}\phi^T + Q \\ x_t^t &= x_t^{t-1} + K_t(y_t - A_t x_t^{t-1}) \\ P_t^t &= [I - K_t A_t]P_t^{t-1} \\ K_t &= P_t^{t-1}A_t^T[A_t P_t^{t-1}A_t^T + R]^{-1} \end{aligned}$$

Some sums:

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \quad \text{for } |a| < 1$$

$$\sum_{n=0}^{\infty} a^n \cos(bn) = \frac{1 - a\cos b}{1 + a^2 - 2a\cos b} \quad \text{for } |a| < 1$$

$$\sum_{n=0}^{\infty} a^n \sin(bn) = \frac{a\sin b}{1 + a^2 - 2a\cos b} \quad \text{for } |a| < 1$$