TMA4285
Tidsrekkemodeller

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Løsningsforslag - Øving 2, 2005

Exercise 1.15

a)

Since s_t has period 12

$$\nabla_{12} X_t = \nabla_{12}(a + bt + s_t + Y_t) = 12b + Y_t - Y_{t-12}$$

so that

$$W_t := \nabla\nabla_{12} X_t = Y_t - Y_{t-1} - Y_{t-12} - Y_{t-13}.$$

Then $\mathrm{E}[W_t] = 0$ and

$$\begin{aligned}\mathrm{Cov}[W_{t+h}, W_t] &= \mathrm{Cov}[Y_{t+h} - Y_{t+h-1} - Y_{t+h-12} - Y_{t+h-13}, Y_t - Y_{t-1} - Y_{t-12} - Y_{t-13}] \\ &= 4\gamma(h) - 2\gamma(h-1) - 2\gamma(h+1) + \gamma(h-11) + \gamma(h+11) - 2\gamma(h-12) \\ &\quad - 2\gamma(h+12) + \gamma(h+13) + \gamma(h-13)\end{aligned}$$

where $\gamma(\cdot)$ is the ACVF of Y_t. Since $\mathrm{E}[W_t]$ and $\mathrm{Cov}[W_{t+h}, W_t]$ are independent of t, W_t is stationary. Also note that $\nabla_{12} X_t$ is stationary.

b)

Using $X_t = (a + bt)s_t + Y_t$ it is obtained that

$$\nabla_{12} X_t = bts_t - b(t-12)s_{t-12} + Y_t - Y_{t-12} = 12bs_{t-12} + Y_t - Y_{t-12}.$$

Now let $U_t = \nabla_{12}^2 X_t = Y_t - 2Y_{t-12} + Y_{t-24}$. Then $\mathrm{E}[U_t] = 0$ and

$$\begin{aligned}\mathrm{Cov}[U_{t+h}, U_t] &= \mathrm{Cov}[Y_{t+h} - 2Y_{t+h-12} + Y_{t+h-24}, Y_t - 2Y_{t-12} + Y_{t-24}] \\ &= 6\gamma(h) - 4\gamma(h+12) - 4\gamma(h-12) + \gamma(h+2) + \gamma(h-24),\end{aligned}$$

which is independent of t. Hence U_t is stationary.

Exercise 2.1

$S(a,b) = E[(X_{n+h} - aX_n - b)^2]$ to be minimized wrt a and b. Now

$S(a,b) = E[\big((X_{n+h}-\mu) - a(X_n-\mu) - b - a\mu + \mu\big)] = \gamma(0) + a^2\gamma(0) + (b+a\mu-\mu)^2 - 2a\gamma(h)$

This gives

$$\frac{\partial S}{\partial a} = 2a\gamma(0) + 2\mu(b + a\mu - \mu) - 2\gamma(h)$$
$$\frac{\partial S}{\partial b} = 2(b + a\mu - \mu)$$

S is clearly minimized wrt b when for $b = \mu(1-a)$. Substituting this value into $\frac{\partial S}{\partial a}$ and equating to zero leads to the result

$$a = \frac{\gamma(h)}{\gamma(0)} = \rho(h)$$

Hence, $S(a,b)$ is minimized when

$$a = \rho(h), \quad b = \mu(1-\rho(h))$$

The BLP (best linear predictor) of X_{n+h} in terms of X_n is therefore $\mu + \rho(h)(X_n - \mu)$.

Exercise 2.3

a)

$$X_t = Z_t + 0.3Z_{t-1} - 0.4Z_{t-2}$$

$$\begin{aligned}
\gamma(0) &= 1 + 0.3^2 + 0.4^2 = 1.25 \\
\gamma(1) &= 0.3 - 0.4 \cdot 0.3 = 0.18 \\
\gamma(2) &= -0.4 \\
\gamma(h) &= 0, \quad h > 2 \\
\gamma(-h) &= \gamma(h)
\end{aligned}$$

b)

$$Y_t = \tilde{Z}_t - 1.2\tilde{Z}_{t-1} - 1.6\tilde{Z}_{t-2}$$

$$\begin{aligned}
\gamma(0) &= 0.25(1 + 1.2^2 + 1.6^2) = 1.25 \\
\gamma(1) &= 0.25(-1.2 + 1.6 \cdot 1.2) = 0.18 \\
\gamma(2) &= -1.6 \cdot 0.25 = -0.4 \\
\gamma(h) &= 0, \quad h > 2 \\
\gamma(-h) &= \gamma(h)
\end{aligned}$$

That is, we obtain the same ACVF as in a).

Exercise 2.5

$\sum_{j=1}^{\infty} \theta^j X_{n-j}$ converges absolutely (with probability 1) since

$$\begin{aligned}
E[\sum_{j=1}^{\infty} |\theta|^j |X_{n-j}|] &\leq \sum_{j=1}^{\infty} |\theta|^j E[|X_{n-j}|] \\
&\leq \sum_{j=1}^{\infty} |\theta|^j \sqrt{\gamma(0) + \mu^2} \quad \text{by Cauchy-Schwartz inequality} \\
&< \infty \quad \text{since} |\theta| < 1
\end{aligned}$$

That is, $\sum_{j=1}^{\infty} |\theta|^j |X_{n-j}| < \infty$ with probability 1.

Mean square convergence of $S_m = \sum_{j=1}^{m} \theta^j X_{n-j}$ as $m \to \infty$ can be verified by invoking Cauchy's criterion. For $m > k$

$$\begin{aligned}
E[|S_m - S_k|^2] &= E[(\sum_{j=k+1}^{m} \theta^j X_{n-j})^2] \\
&= \sum_{i=k+1}^{m} \sum_{j=k+1}^{m} \theta^{i+j} E[X_{n-i} X_{n-j}]
\end{aligned}$$

$$\begin{aligned}
E[|S_m - S_k|^2] &= E[(\sum_{j=k+1}^{m} \theta^j X_{n-j})^2] = \sum_{i=k+1}^{m} \sum_{j=k+1}^{m} \theta^{i+j} E[X_{n-i} X_{n-j}] \\
&= \sum_{i=k+1}^{m} \sum_{j=k+1}^{m} \theta^{i+j} \big(\gamma(i-j) + \mu^2\big) \\
&\leq \sum_{i=k+1}^{m} \sum_{j=k+1}^{m} |\theta|^{i+j} \big(\gamma(0) + \mu^2\big) = \big(\gamma(0) + \mu^2\big) \Big(\sum_{j=k+1}^{m} |\theta|^j \Big)^2 \\
&\to 0 \quad \text{as} \quad k, m \to \infty
\end{aligned}$$

since $\sum_{j=1}^{\infty} |\theta|^j < \infty$. Hence, by Cauchy's mutual convergence criterion, mean square convergence is guaranteed.

Exercise 2.7

$$\begin{aligned}
\frac{1}{1-\phi z} &= \frac{-\frac{1}{\phi z}}{1-\frac{1}{\phi z}} \\
&= -\frac{1}{\phi z}\Big(1 + \frac{1}{\phi z} + \frac{1}{(\phi z)^2} + \dots\Big) \\
&= -\sum_{j=1}^{\infty} (\phi z)^{-j}
\end{aligned}$$

since $|\phi z| > 1$.