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Exercise 3.10

i) From ITSM we find

$$\hat{\gamma}(0) = 676789 = \frac{\hat{\sigma}^2}{1 - \hat{\phi}^2}$$

$$\hat{\gamma}(1) = 676789 \cdot 0.7323 = 495613 = \frac{\hat{\sigma}^2 \hat{\phi}}{1 - \hat{\phi}^2}$$

Solving for $\hat{\sigma}^2$ and $\hat{\phi}$, we obtain $\hat{\sigma}^2 = 313852$ and $\hat{\phi} = 0.7323$. ITSM gives the sample mean as $\hat{\mu} = 4503$, which leads to the model

$$Y_t = 0.7323 Y_{t-1} + Z_t$$

where $Z_t \sim \text{WN}(0, 313852)$ and $Y_t = X_t - 4503$

ii) The BLP (best linear predictor) of Y_{31} is

$$\hat{Y}_{31} = \hat{\phi} Y_{30} = 0.7323\,(3885 - 4503) = -452$$

where the last observation (1980) is $X_{30} = 3885$. Hence, $\hat{X}_{31} = \hat{Y}_{31} + \hat{\mu} = 4051$. The mean square prediction error is $E[(Y_{31} - \hat{Y}_{31})^2] = E[Z_{31}^2] = \sigma^2$, which we estimate by $\hat{\sigma}^2 = 313852$.

iii) 95% prediction error bounds are estimated as

$$4051 \pm 1.96\sqrt{313852} = 4051 \pm 1098 = 2953,\ 5149$$

Exercise 3.11

The PACF at lag 2, ϕ_{22}, of the given MA(1) process is determined from the BLP P_2X_3 of X_3 in terms of X_2 and X_1 by the relation

$$P_2X_3 = \phi_{21}X_2 + \phi_{22}X_1$$

where $Z_t \sim \mathrm{WN}(0, \sigma^2)$.

The value of ϕ_{22} was calculated in Exercise 2.21(a). It was found that

$$\phi_{22} = a_2 = \frac{-\theta^2}{(1+\theta^2)^2 - \theta^2} = \frac{-\theta^2}{1+\theta^2+\theta^4}$$

Exercise 5.1

According to the large sample distribution of Yule-Walker estimators, we have that

$$\begin{bmatrix} \hat{\phi}_1 - \phi_1 \\ \hat{\phi}_2 - \phi_2 \end{bmatrix} \xrightarrow{d} N\big(0, \frac{\sigma^2}{n}\Gamma_2^{-1}\big)$$

where

$$\hat{\Gamma}_2 = \begin{bmatrix} \hat{\gamma}(0) & \hat{\gamma}(1) \\ \hat{\gamma}(1) & \hat{\gamma}(0) \end{bmatrix} = \begin{bmatrix} 1382.2 & 1114.4 \\ 1114.4 & 1382.2 \end{bmatrix}$$

This gives

$$\hat{\Gamma}_2^{-1} = \begin{bmatrix} 0.002067 & -0.001667 \\ -0.001667 & 0.002067 \end{bmatrix}$$

The estimates $\hat{\phi}_1$ and $\hat{\phi}_2$ are obtained from the equation

$$\begin{bmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \end{bmatrix} = \hat{\Gamma}_2^{-1} \begin{bmatrix} \hat{\gamma}(1) \\ \hat{\gamma}(2) \end{bmatrix} = \begin{bmatrix} 1.318 \\ -0.634 \end{bmatrix}$$

while $\hat{\sigma}^2$ is given by the equation

$$\hat{\sigma}^2 = \hat{\gamma}(0) - [1.318, -0.634] \begin{bmatrix} 1114.4 \\ 591.73 \end{bmatrix} = 1382.2 - 1093.0 = 289.2$$

Estimated standard deviation of $\hat{\phi}_1$:

$$\frac{\hat{\sigma}}{\sqrt{n}} \sqrt{\big(\hat{\Gamma}_2^{-1}\big)_{11}} = 0.0773$$

Estimated standard deviation of $\hat{\phi}_2$:

$$\frac{\hat{\sigma}}{\sqrt{n}}\sqrt{\left(\hat{\Gamma}_2^{-1}\right)_{22}} = 0.0773$$

Approximate 95% confidence bounds

$$\phi_1 \;:\; 1.318 \pm 1.96 \times 0.0773 = 1.166, 1.470$$

$$\phi_2 \;:\; -0.634 \pm 1.96 \times 0.0773 = -0.786, -0.482$$

Exercise 5.2

(a) Using the DL algorithm it is obtained that

$$\begin{aligned}
\hat{\phi}_{11} &= \hat{\rho}(1) = \frac{\hat{\gamma}(1)}{\hat{\gamma}(0)} = 0.80625 \\
\hat{v}_1 &= \hat{\gamma}(0)\big(1 - \hat{\rho}(1)^2\big) = 483.72 \\
\hat{\phi}_{22} &= \frac{\hat{\gamma}(2) - \hat{\phi}_{11}\,\hat{\gamma}(1)}{\hat{v}_1} = -0.63412 \\
\hat{\phi}_{21} &= \hat{\phi}_{11}(1 - \hat{\phi}_{22}) = 1.3175 \\
\hat{v}_2 &= \hat{v}_1\big(1 - \hat{\phi}_{22}^2\big) = 289.21 \\
\hat{\phi}_{33} &= \frac{\hat{\gamma}(3) - \hat{\phi}_{21}\,\hat{\gamma}(2) - \hat{\phi}_{22}\,\hat{\gamma}(1)}{\hat{v}_2} = 0.08047 \\
\hat{\phi}_{32} &= \hat{\phi}_{22} - \hat{\phi}_{33}\hat{\phi}_{21} = -0.7401 \\
\hat{\phi}_{31} &= \hat{\phi}_{21} - \hat{\phi}_{33}\hat{\phi}_{22} = 1.3685 \\
\hat{v}_3 &= \hat{v}_2\big(1 - \hat{\phi}_{33}^2\big) = 287.34
\end{aligned}$$

(b) If the data are from an AR(2)-process, $\phi_{33} = 0$ and $\hat{\phi}_{33}$ is an observation from $N(0, 1/100)$. It is seen that $|\hat{\phi}_{33}| = 0.08047 < 1.96/10$. Hence, we would not reject the zero hypothesis $H_0 : \phi_{33} = 0$ at the significance level 0.05.

Exercise 5.5

10 series of 200 observations of an MA(1) process with $\theta = \theta_1 = 0.6$ and white noise variance equal to 1.0, are generated by using for example ITSM. For the moment estimate of θ, the following equation is used

$$\hat{\theta} = \left(1 - \sqrt{1 - 4\hat{\rho}(1)^2}\right) / \left(2\hat{\rho}(1)\right)$$

which is valid for $\hat{\rho}(1) \leq 0.5$. The moment estimates are conveniently calculated by pasting the values for $\hat{\rho}(1)$ into f.ex. Excel.

The Innovations and MLE estimates are obtained directly from ITSM for each data series. The sample mean and sample variance for each type of estimator can also be calculated by using f.ex. Excel.

Obtained results for the seed numbers chosen.

Simulation	rho(1)hat	thetahat	Moments	Innov. Alg.	MLE
1	0.3901	0.479966	0.48	0.592	0.6686
2	0.4001	0.500208	0.5002	0.5909	0.6135
3	0.4611	0.665026	0.665	0.5649	0.557
4	0.4569	0.649853	0.6499	0.6567	0.61
5	0.5489	#NUM!	1	0.755	0.6771
6	0.464	0.676099	0.6761	0.548	0.4859
7	0.4274	0.562754	0.5628	0.5284	0.5521
8	0.4238	0.553757	0.5538	0.6318	0.6378
9	0.3825	0.46532	0.4653	0.5324	0.5741
10	0.3611	0.426912	0.4269	0.5103	0.5581
	Sample variance =		0.027606	0.005475	0.003502
	Sample mean =		0.598	0.59104	0.59342
	Asymptotic theory:		0.023704	0.005	0.0032

The MLE is clearly best in terms of sample variance, The moment estimator is worst. The empirical means and variances of the estimators are in good agreement with the asymptotic theory.