Norges teknisk–
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English

Contact person during the exam:
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EXAM TMA4285 TIME SERIES MODELS

17. December 2005
Time: 09:00–13:00

Allowable aids: *Tabeller og formler i statistikk*, Tapir Forlag
K. Rottmann: *Matematisk formelsamling*
Kalkulator HP30S
One yellow stamped A5 sheet with own formulas and notes.

The result of the exam available 14. January 2005

NB: All answers must be justified.

Notation used in this problem set:

- Z_t is white noise with variance σ^2, that is, $Z_t \sim \text{WN}(0, \sigma^2)$.
- B is *backshift*-operator, such that $B^j X_t \equiv X_{t-j}$, $j \in \mathbf{Z} = \{0, \pm 1, \pm 2, \ldots\}$
- ACVF = autocovariance function, ACF = autocorrelation function.

Problem 1

Assume that the time series X_t is an ARMA$(2,1)$ process defined by

$$\phi(B)\, X_t = \theta(B)\, Z_t\,; \quad t \in \mathbf{Z} \tag{1}$$

where the AR polynomial $\phi(z) = 1 - \phi^2\, z^2$, and the MA polynomial $\theta(z) = 1 + \theta\, z$.

a) What are the requirements that the parameters ϕ and θ must satisfy for X_t to be an ARMA$(2,1)$ process?

b) What does it mean that X_t is a causal time series, and how can that be expressed? What about invertibility?

Which additional requirements must ϕ and θ satisfy for X_t to be a causal and invertible time series?

For the remaining part of this exam problem it is assumed that $0 < |\phi| < 1$.

c) Determine the ACVF $\gamma(h)$ of X_t. Show first that

$$\gamma(0) = \sigma^2 \frac{1+\theta^2}{1-\phi^4} \tag{2}$$

and

$$\gamma(1) = \sigma^2 \frac{\theta}{1-\phi^2} \tag{3}$$

d) Show that for $\theta = 0$, the following equation holds

$$\gamma(2k) = \frac{\sigma^2}{1-\phi^4}\, \phi^{2|k|}\,, \quad k \in \mathbf{Z} \tag{4}$$

What is $\gamma(2k+1)$ for $k \in \mathbf{Z}$?

Problem 2

Given the 'observations' $X_1, X_2, \ldots, X_n$ from a stationary time series X_t, then the ACVF $\gamma(h)$ and the ACF $\rho(h)$ can be estimated as follows $(h = 0, 1, 2, \ldots)$:

$$\hat{\gamma}(h) = \frac{1}{n} \sum_{t=1}^{n-h} (X_{t+h} - \bar{X}_n)(X_t - \bar{X}_n), \tag{5}$$

and

$$\hat{\rho}(h) = \frac{\hat{\gamma}(h)}{\hat{\gamma}(0)}, \tag{6}$$

which can be used if h is not too big compared with n, f.eks. $0 \le h < n/4$. The following result is now assumed: For large n the estimator $\hat{\boldsymbol{\rho}} = \big(\hat{\rho}(1), \hat{\rho}(2), \ldots, \hat{\rho}(k)\big)'$ will be approximately normally distributed $N(\boldsymbol{\rho}, \frac{1}{n}W)$, where $\boldsymbol{\rho} = \big(\rho(1), \rho(2), \ldots, \rho(k)\big)'$ and $W = (w_{ij})$ is a $k \times k$ covariance matrix given by Bartlett's formula

$$w_{ij} = \sum_{l=1}^{\infty} \{\rho(l+i) + \rho(l-i) - 2\rho(i)\,\rho(l)\} \cdot \{\rho(l+j) + \rho(l-j) - 2\rho(j)\,\rho(l)\} \tag{7}$$

Assume that for the observations $x_1, x_2, \ldots, x_{100}$, we have found that $\hat{\rho}(1) = 0.438$ og $\hat{\rho}(2) = 0.145$.

a) Assume that the observations have been generated by an AR(1) process: $X_t = \phi\, X_{t-1} + Z_t$, where $|\phi| < 1$. Set up the relation between $\rho(h)$ and ϕ, and show that

$$\mathrm{Var}[\hat{\rho}(1)] \approx \frac{1}{n}\big(1 - \phi^2\big)\,. \tag{8}$$

Determine a corresponding expression for $\mathrm{Var}[\hat{\rho}(2)]$. Hint: Use e.g. the observed value $\hat{\rho}(2) = 0.145$ to justify the approximations.

Then construct an approximate 95% confidence interval for both $\rho(1)$ and $\rho(2)$. Use these two confidence intervals to discuss whether the observations are consistent with an AR(1) model with $\phi = 0.8$.

b) Assume that the observations are generated by an MA(1) process. Construct an approximate 95% confidence interval for both $\rho(1)$ and $\rho(2)$. Use the two confidence intervals to discuss whether the observations are consistent with an MA(1) model with $\theta = 0.6$.

GIVEN: The 95% confidence limits for an $N(0,1)$ variable are ± 1.96.

Problem 3

a) Assume that the time series X_t, $t \in \mathbf{Z}$, is (weakly) stationary. Show that $\nabla^k X_t = (1-B)^k X_t$ is also a stationary time series for any $k = 1, 2, \ldots$.

b) Let the time series Y_t be defined by the equation

$$Y_t = \beta_0 + \beta_1\, t + \beta_2\, t^2 + X_t\,, \quad t \in \mathbf{Z},\ \beta_2 > 0, \tag{9}$$

where X_t is stationary. Does there exist an integer k_0 such that $\nabla^k Y_t$ is stationary for $k \ge k_0$, while $\nabla^k Y_t$ is not stationary for $k < k_0$. What is then k_0?