Norwegian University of Science and Technology
Department of Mathematical Sciences

English

Contact person during the exam:
Arvid Næss 73 59 70 53/ 99 53 83 50

EXAM TMA4285 TIME SERIES MODELS

11. December 2008
Time: 09:00–13:00

Permitted aids: *Tabeller og formler i statistikk*, Tapir Forlag
K. Rottmann: *Matematisk formelsamling*
Calculator HP30S or CITIZEN SR-270X
One yellow, stamped A5 sheet with own formulas and notes.

The results of the exam available 10. January 2008

NB: All answers must be justified.

Notation used in this problem set:

- Z_t is white noise with variance σ^2, that is, $Z_t \sim \mathrm{WN}(0, \sigma^2)$.
- B is *backshift*-operator, such that $B^j X_t \equiv X_{t-j}$, $j \in \mathbf{Z} = \{0, \pm 1, \pm 2, \ldots\}$
- ACVF = autocovariance function, ACF = autocorrelation function.
- IID = independent, identically distributed.
- N(0,1) = normally distributed with mean value 0 and variance 1.0.

Problem 1

You will find the figures for this problem at the end of the problem set.

a) Assume that the time series X_t, $t \in \mathbf{Z}$, is an AR(1) process given by $X_t + \phi\, X_{t-1} = Z_t$, where Z_t denotes white noise. In Fig. 1 are shown the ACF for 4 different values of the parameter ϕ. Determine these four values in the following order: 1) Upper left hand figure, 2) Upper right hand figure, 3) Lower left hand figure, and 4) Lower right hand figure.

b) Assume that the time series Y_t, $t \in \mathbf{Z}$, is an MA(q) process. In Fig. 2 are shown plots of Y_t versus Y_{t-k} for $k = 1, 2, 3$. Look very carefully at these plots and try to decide what the value of q is.

c) Assume that an observed time series X_t representing the demand of electricity over a period of more than 30 years looks like the one shown in Fig. 3. If you wanted to try to fit an ARMA model to this time series, suggest the first steps you would take in your efforts to make such a fit. Explain why.

Problem 2

Assume that the time series X_t is an ARMA(2, 1) process defined by

$$\phi(B)\, X_t = \theta(B)\, Z_t\,; \quad t \in \mathbf{Z} \tag{1}$$

where the AR polynomial $\phi(z) = 1 - z + \phi^2\, z^2$, and the MA polynomial $\theta(z) = 1 + \theta\, z$.

a) What are the requirements that the parameters ϕ and θ must satisfy for X_t to be a (stationary) ARMA(2, 1) process? Hint: It may help you to show that for $\phi \neq 0$, the AR polynomial has the roots

$$z_{1,2} = \frac{2}{1 \pm \sqrt{1 - 4\phi^2}}.$$

b) Which additional requirements must ϕ and θ satisfy for X_t to be a causal and invertible time series?

For the remaining part of this exam problem it is assumed that $0 < |\phi| < 1$ and $\theta = 0$. The following result is also cited: For an AR(2) process the ACVF can be expressed as $\gamma(h) = c_1 z_1^{-h} + c_2 z_2^{-h}$ for $h \geq 0$ when $z_1 \neq z_2$, where $\phi(z_j) = 0\,,\;\; j = 1, 2$; while $\gamma(h) = (c_1 + c_2 h) z_1^{-h}$ for $h \geq 0$ when $z_1 = z_2$.

c) Depending on the value of ϕ, $\gamma(h)$ will display qualitatively different behaviour. Without determining the constants c_1 and c_2 explicitly in terms of ϕ and σ, write down the expressions for $\gamma(h)$ as it depends on ϕ. Make sure that $\gamma(h)$ is always a real function of h.

d) Determine the explicit expression for $\gamma(h)$ for $\phi^2 = 1/2$, and show that the ACF is given as follows,

$$\rho(h) = \frac{\cos\left(\frac{\pi}{4}|h| + b\right)}{2^{|h|/2}\cos(b)}\,, \quad h \in \mathbf{Z} \tag{2}$$

where $\tan b = 2/7$. (Hint: $\cos t = (e^{\mathrm{i}t} + e^{-\mathrm{i}t})/2$, $\mathrm{i} = \sqrt{-1}$.)

Problem 3

An ARCH(1) process X_t is given as,

$$X_t = \sqrt{H_t}\,\varepsilon_t\,, \;\; \varepsilon_t \sim \text{IID N}(0,1)\,, \tag{3}$$

where

$$H_t = \alpha_0 + \alpha_1 X_{t-1}^2\,, \;\; (\alpha_0 > 0, \alpha_1 > 0). \tag{4}$$

It is assumed that $0 < \alpha_1 < 1$. It can then be shown that

$$H_t = \alpha_0\big(1 + \sum_{j=1}^{\infty} \alpha_1^j \varepsilon_{t-1}^2 \cdot \ldots \cdot \varepsilon_{t-j}^2\big). \tag{5}$$

a) Argue why ε_t and H_t are independent random variables for each t.

Determine the expressions for $\mathrm{E}[X_t]$, $\mathrm{E}[X_t^2]$.

It turns out that $\mathrm{E}[X_t^4]$ does not exist for every α_1 satisfying $0 < \alpha_1 < 1$. Find the expression for $\mathrm{E}[X_t^4]$ and give the values of α_1 for its existence. ($\mathrm{E}[\varepsilon_t^4] = 3$.)

b) Let the process η_t be defined as follows, assuming that $\mathrm{E}[X_t^4] < \infty$,

$$\eta_t = X_t^2 - H_t = (\varepsilon_t^2 - 1)H_t. \tag{6}$$

Show that $\eta_t \sim \mathrm{WN}(0, \sigma_0^2)$, and determine σ_0^2.

c) Establish an ARMA(p, q) model for X_t^2, and identify p and q.

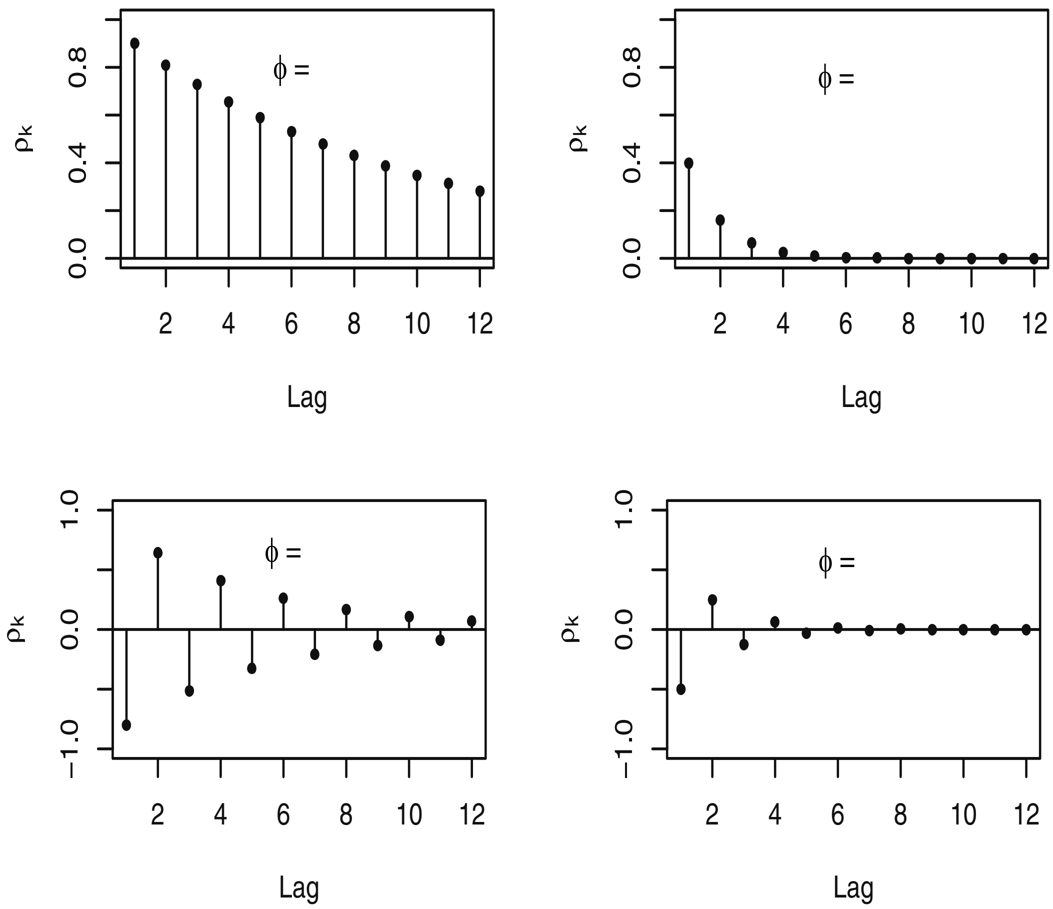

Figure 1: ACF for different values of ϕ

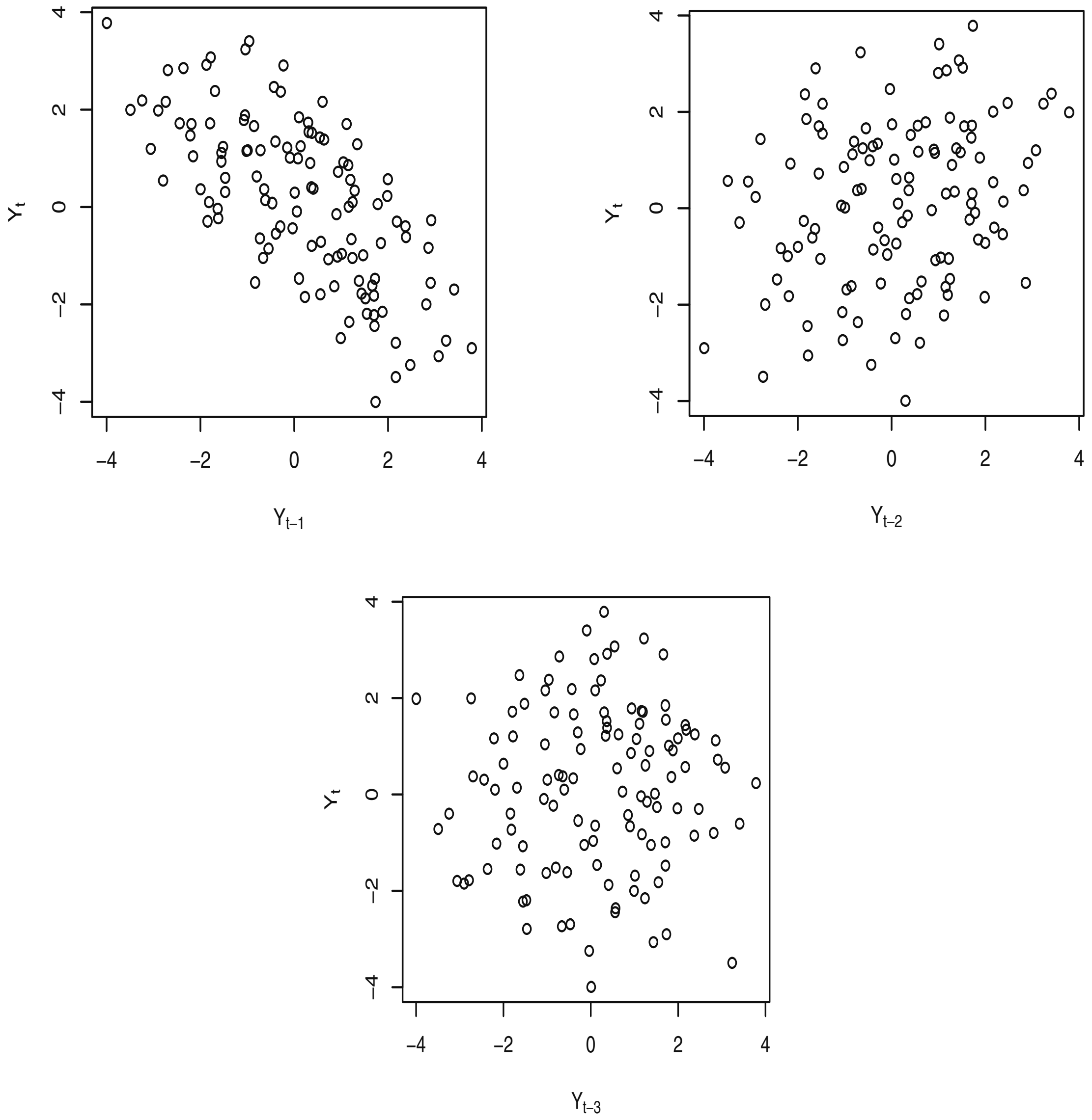

Figure 2:

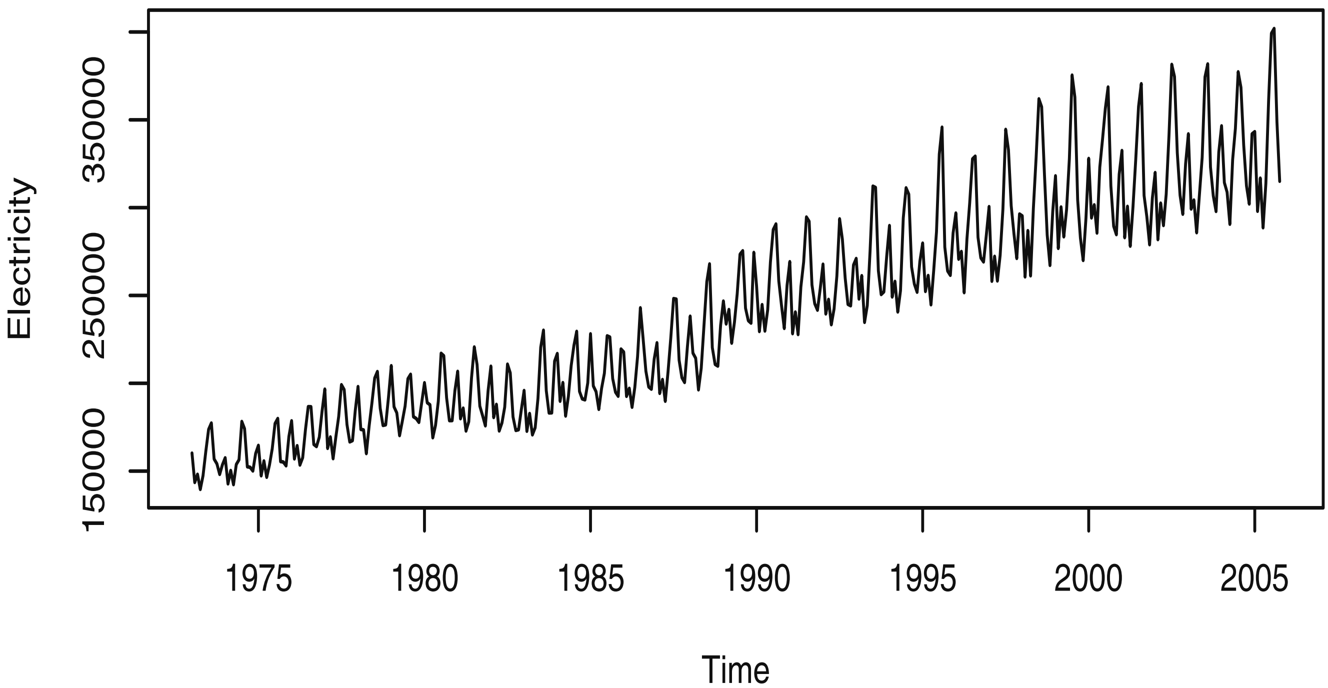

Figure 3: Demand for electricity.

Norges teknisk–
naturvitenskapelige universitet
Institutt for matematiske fag

English

SOLUTION SKETCH FOR

TMA4285 TIME SERIES MODELS

11. December 2008
Time: 09:00–13:00

Problem 1

a) The ACF $\rho(k) = \rho_k$ for the given AR(1) process is given as $\rho_k = (-\phi)^k$ for $k = 0, 1, 2, \ldots$. From the figures one may therefore read off the values of ϕ: 1) $\phi = -0.9$, 2) $\phi = -0.4$, 3) $\phi = 0.8$, 4) $\phi = 0.5$

b) The plot of Y_t versus Y_{t-1} displays a significant negative correlation, while the plot of Y_t versus Y_{t-2} indicates a weak positive correlation (draw a vertical and horizontal line through the origin and study each of the quadrants). No such trend can be detected in the plot of Y_t versus Y_{t-3}. This indicates that the data is generated by an MA(2) model.

c) The recorded time series displays both increasing variability (variance), trend and seasonality. Therefore, in order to stabilize the variability, a transformation of the data is suggested. This could be done by the Cox-Box transformation, a log transformation is a typical choice. After an appropriate transformation, the trend is removed by differencing. Here one differencing would seem sufficient. Finally, the seasonal component may be removed by s-differencing, where s denotes the identified seasonal period. The residual process after these operations can then be studied for a possible ARMA model fit.

Problem 2

a) For X_t to be an ARMA(2,1) process it has to be stationary, and the AR and MA polynomials cannot have common roots. Stationarity is guaranteed by $\phi(z) \neq 0$ for $|z| = 1$ ($z \in \mathbf{C}$ = the complex numbers). That is, the roots of the AR polynomial $\phi(z) = 1 - z + \phi^2 z^2$ cannot lie on the unit circle. For $\phi = 0$, obviously $\phi(z) = 0$ for $z = 1$.

Hence, a necessary condition for X_t to be stationary is that $\phi \neq 0$. For $\phi \neq 0$, i.e. $\phi^2 > 0$, the roots of $\phi(z)$ are obtained as,

$$z^{\pm} = \frac{1 \mp \sqrt{1-4\phi^2}}{2\phi^2} = \frac{2}{1 \pm \sqrt{1-4\phi^2}}.$$

Can $|z^{\pm}| = 1$ for some value of $\phi \neq 0$? It is seen that $z^{\pm}$ are real for $0 < \phi^2 \leq 1/4$, and that $|z^{\pm}| > 1$. For $\phi^2 > 1/4$, there will be two complex conjugate roots $z^{\pm} = 2/(1 \pm \mathrm{i}\sqrt{4\phi^2 - 1})$ ($\mathrm{i} = \sqrt{-1}$). It follows that $|z^{\pm}| = 1$ if $4 = 1 + (4\phi^2 - 1) = 4\phi^2$, which is satisfied for $\phi = \pm 1$. Conclusion: X_t is stationary for $\phi \notin \{0, \pm 1\}$

The requirement of no common roots need only be checked when $0 < \phi^2 \leq 1/4$. The root of the MA polynomial is $z = -1/\theta$. Hence, there will be no common roots when

$$\theta \neq \frac{-1 \pm \sqrt{1-4\phi^2}}{2}\,, \ \text{ for } \ 0 < \phi^2 \leq 1/4\,,$$

else θ is arbitrary.

b) For X_t to be causal, $|z^{\pm}| > 1$. According to a) we only need to investigate the parameter range $\phi^2 > 1/4$. In this range, it follows that $|z^{\pm}| > 1$ if and only if $\phi^2 < 1$. Hence, X_t is causal if and only if $0 < \phi^2 < 1$.

X_t is invertible if and only if $1 + \theta z \neq 0$ for $|z| \leq 1$, that is, for $|1/\theta| > 1$. Hence, X_t is invertible if and only if $|\theta| < 1$.

c) For the AR(2) process at hand there will be three cases to consider:

1. For $0 < \phi^2 < 1/4$, and $h \in \mathbf{Z}$,

$$\gamma(h) = c_1\Big(\frac{1+\sqrt{1-4\phi^2}}{2}\Big)^{|h|} + c_2\Big(\frac{1-\sqrt{1-4\phi^2}}{2}\Big)^{|h|}\,,$$

   where c_1 and c_2 are two real constants.

2. For $\phi^2 = 1/4$, and $h \in \mathbf{Z}$,

$$\gamma(h) = (c_1 + c_2|h|)2^{-|h|}$$

   where c_1 and c_2 are two real constants.

3. For $1/4 < \phi^2 < 1$, and $h \in \mathbf{Z}$,

$$\gamma(h) = c\Big(\frac{1+\mathrm{i}\sqrt{4\phi^2-1}}{2}\Big)^{|h|} + \overline{c}\Big(\frac{1-\mathrm{i}\sqrt{4\phi^2-1}}{2}\Big)^{|h|}\,,$$

where $c = c_1$ is a complex number in general, and $c_2 = \overline{c}$, which is the complex conjugate of c. This is necessary to make $\gamma(h)$ real. Using polar representation of a complex number, we may write $c = (a/2)\, e^{\mathrm{i}b}$ for suitable real numbers a and b. Also

$$\frac{1 + \mathrm{i}\sqrt{4\phi^2 - 1}}{2} = |\phi|\, e^{\mathrm{i}\theta}\,,$$

where $\tan\theta = \sqrt{4\phi^2 - 1}$. It is then obtained that

$$\gamma(h) = a|\phi|^{|h|}\cos(\theta|h| + b)\,,$$

d) Using the relation $X_t = X_{t-1} - \frac{1}{2}X_{t-2} + Z_t$, and calculating $\mathrm{E}[X_tX_t]$ and $\mathrm{E}[X_tX_{t+1}]$, the following two equations are obtained,

$$\gamma(1) = \frac{1}{4}\gamma(0) + \sigma^2$$

and

$$\gamma(2) = \frac{1}{2}\gamma(1) - \gamma(0) + 2\sigma^2 = -\frac{7}{8}\gamma(0) + \frac{5}{2}\sigma^2$$

From the previous point we know that

$$\gamma(h) = c\Big(\frac{1+\mathrm{i}}{2}\Big)^{|h|} + \overline{c}\Big(\frac{1-\mathrm{i}}{2}\Big)^{|h|}\,,$$

which leads to the equations

$$c\,\frac{1+\mathrm{i}}{2} + \overline{c}\,\frac{1-\mathrm{i}}{2} = \frac{1}{4}(c + \overline{c}) + \sigma^2\,,$$

and

$$c\,\mathrm{i} - \overline{c}\,\mathrm{i} = -\frac{7}{8}(c + \overline{c}) + \frac{5}{2}\sigma^2\,.$$

Solving the equations gives the solution

$$c = \frac{4}{13}(7 + 2\mathrm{i})\,\sigma^2 = \frac{4}{13}\sqrt{53}\,e^{\mathrm{i}\,0.278}\,,$$

where $0.278 = \tan^{-1}(2/7)$. Noting that $(1+\mathrm{i})/2 = (1/\sqrt{2})\,e^{\mathrm{i}\pi/4}$, it follows that

$$\gamma(h) = \frac{8}{13}\sqrt{53}\,\sigma^2\,\Big(\frac{1}{2}\Big)^{|h|/2}\cos(\frac{\pi}{4}|h| + 0.278)\,,$$

The ACF then becomes

$$\rho(h) = \frac{\cos\left(\frac{\pi}{4}|h| + 0.278\right)}{2^{|h|/2}\cos(0.278)}\,,\quad h \in \mathbf{Z} \tag{1}$$

Problem 3

a) Since ε_t is IID, it follows that ε_t is independent of ε_s for every $s < t$. This implies that ε_t is independent of $\varepsilon_{t-1}^2 \cdot \ldots \cdot \varepsilon_{t-j}^2$ for every $j = 1, 2, \ldots$. Except for constants, $H(t)$ consists of a sum of such terms, and therefore ε_t and H_t are independent random variables for each t. Then ε_t and $\sqrt{H_t}$ are also independent random variables for each t. It is then obtained that,

$$\mathrm{E}[X_t] = \mathrm{E}[\varepsilon_t]\,\mathrm{E}[\sqrt{H_t}] = 0\,,$$

and

$$\begin{aligned}\mathrm{E}[X_t^2] &= \mathrm{E}[\varepsilon_t^2\, H_t] = \mathrm{E}[\varepsilon_t^2]\,\mathrm{E}[H_t] = \mathrm{E}[H_t] \\ &= \alpha_0\big(1 + \sum_{j=1}^{\infty} \alpha_1^j \mathrm{E}[\varepsilon_{t-1}^2] \cdot \ldots \cdot \mathrm{E}[\varepsilon_{t-j}^2]\big) = \alpha_0\big(1 + \sum_{j=1}^{\infty} \alpha_1^j\big) = \frac{\alpha_0}{1-\alpha_1}\,.\end{aligned}$$

Since ε_t is IID, it is seen that X_t is strictly stationary. Hence, moments, when they exist, are independent of t. From the equation $X_t^2 = \varepsilon_t^2(\alpha_0 + \alpha_1 X_{t-1}^2)$, it follows that

$$X_t^4 = \varepsilon_t^4\,(\alpha_0^2 + 2\alpha_0\alpha_1 X_{t-1}^2 + \alpha_1^2 X_{t-1}^4)\,,$$

Hence, if $m_4 = \mathrm{E}[X_t^4]$ exists, it must satisfy the equation ($\mathrm{E}[\varepsilon_t^4] = 3$),

$$m_4 = 3\,(\alpha_0^2 + 2\alpha_0\alpha_1 \frac{\alpha_0}{1-\alpha_1} + \alpha_1^2 m_4)\,,$$

This leads to the equation

$$m_4 = \frac{3\,\alpha_0^2(1+\alpha_1)}{(1-\alpha_1)(1-3\alpha_1^2)}\,.$$

This equation can only be satisfied if $0 < \alpha_1^2 < 1/3$, which becomes the condition for finite m_4.

b) By a similar argument as above, it is seen that η_t is a strictly stationary process. By the assumption that $\mathrm{E}[X_t^4] < \infty$, it follows that η_t has finite second order moments. It is therefore a (weakly) stationary process.

$$\mathrm{E}[\eta_t] = \mathrm{E}[X_t^2] - \mathrm{E}[H_t] = 0\,.$$

and for $h \geq 1$,

$$\mathrm{E}[\eta_{t+h}\,\eta_t] = \mathrm{E}[(\varepsilon_{t+h}^2 - 1)(\varepsilon_t^2 - 1)\,H_{t+h}H_t] = \mathrm{E}[(\varepsilon_{t+h}^2 - 1)]\,\mathrm{E}[(\varepsilon_t^2 - 1)\,H_{t+h}H_t] = 0\,.$$

since ε_{t+h} is independent of ε_s for every $s < t+h$, and therefore $\varepsilon_{t+h}^2 - 1$ is independent of $(\varepsilon_t^2 - 1)\,H_{t+h}H_t$. It follows that η_t is white noise with variance

$$\begin{gathered}\sigma_0^2 = \mathrm{E}[\eta_t^2] = \mathrm{E}[(\varepsilon_t^2 - 1)^2\, H_t^2] = \mathrm{E}[(\varepsilon_t^2 - 1)^2]\,\mathrm{E}[H_t^2] = \\ \mathrm{E}[\varepsilon_t^4 - 2\varepsilon_t^2 + 1]\,\frac{\mathrm{E}[X_t^4]}{\mathrm{E}[\varepsilon_t^4]} = \frac{2}{3} m_4 = \frac{2\,\alpha_0^2(1+\alpha_1)}{(1-\alpha_1)(1-3\alpha_1^2)}\end{gathered}$$

c) From the definition of η_t, it follows that

$$X_t^2 = H_t + \eta_t = \alpha_0 + \alpha_1 X_{t-1}^2 + \eta_t. \tag{2}$$

Introducing the process $Y_t = X_t^2 - \alpha_0/(1-\alpha_1)$, it is obtained that,

$$Y_t = \alpha_1 Y_{t-1} + \eta_t. \tag{3}$$

Since $0 < \alpha_1 < 1/\sqrt{3}$, it is seen that $\phi(z) = 1 - \alpha_1 z \neq 0$ for $|z| \leq 1$. It follows that Y_t becomes a causal AR(1) process, and therefore also X_t^2 (in the non-zero mean form).