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English

SOLUTION SKETCH FOR

TMA4285 TIME SERIES MODELS

11. December 2008
Time: 09:00–13:00

Problem 1

a) The ACF $\rho(k) = \rho_k$ for the given AR(1) process is given as $\rho_k = (-\phi)^k$ for $k = 0, 1, 2, \ldots$. From the figures one may therefore read off the values of ϕ: 1) $\phi = -0.9$, 2) $\phi = -0.4$, 3) $\phi = 0.8$, 4) $\phi = 0.5$

b) The plot of Y_t versus Y_{t-1} displays a significant negative correlation, while the plot of Y_t versus Y_{t-2} indicates a weak positive correlation (draw a vertical and horizontal line through the origin and study each of the quadrants). No such trend can be detected in the plot of Y_t versus Y_{t-3}. This indicates that the data is generated by an MA(2) model.

c) The recorded time series displays both increasing variability (variance), trend and seasonality. Therefore, in order to stabilize the variability, a transformation of the data is suggested. This could be done by the Cox-Box transformation, a log transformation is a typical choice. After an appropriate transformation, the trend is removed by differencing. Here one differencing would seem sufficient. Finally, the seasonal component may be removed by s-differencing, where s denotes the identified seasonal period. The residual process after these operations can then be studied for a possible ARMA model fit.

Problem 2

a) For X_t to be an ARMA(2,1) process it has to be stationary, and the AR and MA polynomials cannot have common roots. Stationarity is guaranteed by $\phi(z) \neq 0$ for $|z| = 1$ ($z \in \mathbf{C}$ = the complex numbers). That is, the roots of the AR polynomial $\phi(z) = 1 - z + \phi^2 z^2$ cannot lie on the unit circle. For $\phi = 0$, obviously $\phi(z) = 0$ for $z = 1$.

Hence, a necessary condition for X_t to be stationary is that $\phi \neq 0$. For $\phi \neq 0$, i.e. $\phi^2 > 0$, the roots of $\phi(z)$ are obtained as,

$$z^{\pm} = \frac{1 \mp \sqrt{1-4\phi^2}}{2\phi^2} = \frac{2}{1 \pm \sqrt{1-4\phi^2}}.$$

Can $|z^{\pm}| = 1$ for some value of $\phi \neq 0$? It is seen that $z^{\pm}$ are real for $0 < \phi^2 \leq 1/4$, and that $|z^{\pm}| > 1$. For $\phi^2 > 1/4$, there will be two complex conjugate roots $z^{\pm} = 2/(1 \pm \mathrm{i}\sqrt{4\phi^2 - 1})$ ($\mathrm{i} = \sqrt{-1}$). It follows that $|z^{\pm}| = 1$ if $4 = 1 + (4\phi^2 - 1) = 4\phi^2$, which is satisfied for $\phi = \pm 1$. Conclusion: X_t is stationary for $\phi \notin \{0, \pm 1\}$

The requirement of no common roots need only be checked when $0 < \phi^2 \leq 1/4$. The root of the MA polynomial is $z = -1/\theta$. Hence, there will be no common roots when

$$\theta \neq \frac{-1 \pm \sqrt{1-4\phi^2}}{2}\,, \ \text{ for } \ 0 < \phi^2 \leq 1/4\,,$$

else θ is arbitrary.

b) For X_t to be causal, $|z^{\pm}| > 1$. According to a) we only need to investigate the parameter range $\phi^2 > 1/4$. In this range, it follows that $|z^{\pm}| > 1$ if and only if $\phi^2 < 1$. Hence, X_t is causal if and only if $0 < \phi^2 < 1$.

X_t is invertible if and only if $1 + \theta z \neq 0$ for $|z| \leq 1$, that is, for $|1/\theta| > 1$. Hence, X_t is invertible if and only if $|\theta| < 1$.

c) For the AR(2) process at hand there will be three cases to consider:

1. For $0 < \phi^2 < 1/4$, and $h \in \mathbf{Z}$,

$$\gamma(h) = c_1\Big(\frac{1+\sqrt{1-4\phi^2}}{2}\Big)^{|h|} + c_2\Big(\frac{1-\sqrt{1-4\phi^2}}{2}\Big)^{|h|},$$

   where c_1 and c_2 are two real constants.

2. For $\phi^2 = 1/4$, and $h \in \mathbf{Z}$,

$$\gamma(h) = (c_1 + c_2|h|)2^{-|h|}$$

   where c_1 and c_2 are two real constants.

3. For $1/4 < \phi^2 < 1$, and $h \in \mathbf{Z}$,

$$\gamma(h) = c\Big(\frac{1+\mathrm{i}\sqrt{4\phi^2-1}}{2}\Big)^{|h|} + \overline{c}\Big(\frac{1-\mathrm{i}\sqrt{4\phi^2-1}}{2}\Big)^{|h|},$$

where $c = c_1$ is a complex number in general, and $c_2 = \overline{c}$, which is the complex conjugate of c. This is necessary to make $\gamma(h)$ real. Using polar representation of a complex number, we may write $c = (a/2)\, e^{\mathrm{i}b}$ for suitable real numbers a and b. Also

$$\frac{1 + \mathrm{i}\sqrt{4\phi^2 - 1}}{2} = |\phi|\, e^{\mathrm{i}\theta} \,,$$

where $\tan\theta = \sqrt{4\phi^2 - 1}$. It is then obtained that

$$\gamma(h) = a|\phi|^{|h|} \cos(\theta|h| + b) \,,$$

d) Using the relation $X_t = X_{t-1} - \frac{1}{2}X_{t-2} + Z_t$, and calculating $\mathrm{E}[X_t X_t]$ and $\mathrm{E}[X_t X_{t+1}]$, the following two equations are obtained,

$$\gamma(1) = \frac{1}{4}\gamma(0) + \sigma^2$$

and

$$\gamma(2) = \frac{1}{2}\gamma(1) - \gamma(0) + 2\sigma^2 = -\frac{7}{8}\gamma(0) + \frac{5}{2}\sigma^2$$

From the previous point we know that

$$\gamma(h) = c\Big(\frac{1+\mathrm{i}}{2}\Big)^{|h|} + \overline{c}\Big(\frac{1-\mathrm{i}}{2}\Big)^{|h|} \,,$$

which leads to the equations

$$c\,\frac{1+\mathrm{i}}{2} + \overline{c}\,\frac{1-\mathrm{i}}{2} = \frac{1}{4}(c + \overline{c}) + \sigma^2 \,,$$

and

$$c\,\mathrm{i} - \overline{c}\,\mathrm{i} = -\frac{7}{8}(c + \overline{c}) + \frac{5}{2}\sigma^2 \,.$$

Solving the equations gives the solution

$$c = \frac{4}{13}(7 + 2\mathrm{i})\,\sigma^2 = \frac{4}{13}\sqrt{53}\, e^{\mathrm{i}\,0.278} \,,$$

where $0.278 = \tan^{-1}(2/7)$. Noting that $(1+\mathrm{i})/2 = (1/\sqrt{2})\, e^{\mathrm{i}\pi/4}$, it follows that

$$\gamma(h) = \frac{8}{13}\sqrt{53}\,\sigma^2 \left(\frac{1}{2}\right)^{|h|/2} \cos(\frac{\pi}{4}|h| + 0.278) \,,$$

The ACF then becomes

$$\rho(h) = \frac{\cos\left(\frac{\pi}{4}|h| + 0.278\right)}{2^{|h|/2}\cos(0.278)} \,, \quad h \in \mathbf{Z} \tag{1}$$

Problem 3

a) Since ε_t is IID, it follows that ε_t is independent of ε_s for every $s < t$. This implies that ε_t is independent of $\varepsilon_{t-1}^2 \cdot \ldots \cdot \varepsilon_{t-j}^2$ for every $j = 1, 2, \ldots$. Except for constants, $H(t)$ consists of a sum of such terms, and therefore ε_t and H_t are independent random variables for each t. Then ε_t and $\sqrt{H_t}$ are also independent random variables for each t. It is then obtained that,

$$\mathrm{E}[X_t] = \mathrm{E}[\varepsilon_t]\,\mathrm{E}[\sqrt{H_t}] = 0\,,$$

and

$$\begin{aligned}\mathrm{E}[X_t^2] &= \mathrm{E}[\varepsilon_t^2\, H_t] = \mathrm{E}[\varepsilon_t^2]\,\mathrm{E}[H_t] = \mathrm{E}[H_t] \\ &= \alpha_0\big(1 + \sum_{j=1}^{\infty} \alpha_1^j \mathrm{E}[\varepsilon_{t-1}^2] \cdot \ldots \cdot \mathrm{E}[\varepsilon_{t-j}^2]\big) = \alpha_0\big(1 + \sum_{j=1}^{\infty} \alpha_1^j\big) = \frac{\alpha_0}{1-\alpha_1}\,.\end{aligned}$$

Since ε_t is IID, it is seen that X_t is strictly stationary. Hence, moments, when they exist, are independent of t. From the equation $X_t^2 = \varepsilon_t^2(\alpha_0 + \alpha_1 X_{t-1}^2)$, it follows that

$$X_t^4 = \varepsilon_t^4\,(\alpha_0^2 + 2\alpha_0\alpha_1 X_{t-1}^2 + \alpha_1^2 X_{t-1}^4)\,,$$

Hence, if $m_4 = \mathrm{E}[X_t^4]$ exists, it must satisfy the equation ($\mathrm{E}[\varepsilon_t^4] = 3$),

$$m_4 = 3\,(\alpha_0^2 + 2\alpha_0\alpha_1 \frac{\alpha_0}{1-\alpha_1} + \alpha_1^2 m_4)\,,$$

This leads to the equation

$$m_4 = \frac{3\,\alpha_0^2(1+\alpha_1)}{(1-\alpha_1)(1-3\alpha_1^2)}\,.$$

This equation can only be satisfied if $0 < \alpha_1^2 < 1/3$, which becomes the condition for finite m_4.

b) By a similar argument as above, it is seen that η_t is a strictly stationary process. By the assumption that $\mathrm{E}[X_t^4] < \infty$, it follows that η_t has finite second order moments. It is therefore a (weakly) stationary process.

$$\mathrm{E}[\eta_t] = \mathrm{E}[X_t^2] - \mathrm{E}[H_t] = 0\,.$$

and for $h \geq 1$,

$$\mathrm{E}[\eta_{t+h}\,\eta_t] = \mathrm{E}[(\varepsilon_{t+h}^2 - 1)(\varepsilon_t^2 - 1)\,H_{t+h}H_t] = \mathrm{E}[(\varepsilon_{t+h}^2 - 1)]\,\mathrm{E}[(\varepsilon_t^2 - 1)\,H_{t+h}H_t] = 0\,.$$

since ε_{t+h} is independent of ε_s for every $s < t+h$, and therefore $\varepsilon_{t+h}^2 - 1$ is independent of $(\varepsilon_t^2 - 1)\,H_{t+h}H_t$. It follows that η_t is white noise with variance

$$\begin{aligned}\sigma_0^2 = \mathrm{E}[\eta_t^2] = \mathrm{E}[(\varepsilon_t^2 - 1)^2\,H_t^2] = \mathrm{E}[(\varepsilon_t^2 - 1)^2]\,\mathrm{E}[H_t^2] = \\ \mathrm{E}[\varepsilon_t^4 - 2\varepsilon_t^2 + 1]\,\frac{\mathrm{E}[X_t^4]}{\mathrm{E}[\varepsilon_t^4]} = \frac{2}{3}m_4 = \frac{2\,\alpha_0^2(1+\alpha_1)}{(1-\alpha_1)(1-3\alpha_1^2)}\end{aligned}$$

c) From the definition of η_t, it follows that

$$X_t^2 = H_t + \eta_t = \alpha_0 + \alpha_1 X_{t-1}^2 + \eta_t. \tag{2}$$

Introducing the process $Y_t = X_t^2 - \alpha_0/(1-\alpha_1)$, it is obtained that,

$$Y_t = \alpha_1 Y_{t-1} + \eta_t. \tag{3}$$

Since $0 < \alpha_1 < 1/\sqrt{3}$, it is seen that $\phi(z) = 1 - \alpha_1 z \neq 0$ for $|z| \leq 1$. It follows that Y_t becomes a causal AR(1) process, and therefore also X_t^2 (in the non-zero mean form).