Time Series Models 2008

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**Solutions Chapter 1, 2008**

## Exercise 1.1

### a)

Let $S(c) = E[(Y-c)^2]$. Then
$$S(c) = E(Y^2) - 2cE(Y) + c^2$$
This gives
$$\frac{dS}{dc} = -2E(Y) + 2c = 0$$
for $c = E(Y)$, which leads to a global minimum since $\frac{d^2S}{dc^2} = 2 > 0$ for all $c$.

### b)

$$\begin{aligned}
E[\big(Y-f(X)\big)^2|X] &= E[\big(Y-E(Y|X)+E(Y|X)-f(X)\big)^2|X] = \\
E[\big(Y-E(Y|X)\big)^2|X] &+ 2E[\big(Y-E(Y|X)\big)\big(E(Y|X)-f(X)\big)|X] + E[\big(E(Y|X)-f(X)\big)^2|X] = \\
E[\big(Y-E(Y|X)\big)^2|X] &+ 2\big(E(Y|X)-f(X)\big)E[\big(Y-E(Y|X)\big)|X] + E[\big(E(Y|X)-f(X)\big)^2|X] = \\
E[\big(Y-E(Y|X)\big)^2|X] &+ E[\big(E(Y|X)-f(X)\big)^2|X] \geq E[\big(Y-E(Y|X)\big)^2|X]
\end{aligned}$$

because $E(Y|X)$ is a function of $X$ and $E(g(X)Y|X) = g(X)E(Y|X)$ for any function $g$ such that $E(g(X)Y)$ exists.

It follows that
$$E[\big(Y-E(Y|X)\big)^2|X] \leq E[\big(Y-f(X)\big)^2|X]$$
for any function $f$. Hence $E[\big(Y-f(X)\big)^2|X]$ is minimized when $f(X) = E(Y|X)$.

### c)

Since
$$E[\big(Y-E(Y|X)\big)^2] = E\Big(E[\big(Y-E(Y|X)\big)^2|X]\Big) \leq E\Big(E[\big(Y-f(X)\big)^2|X]\Big) = E[\big(Y-f(X)\big)^2]$$
it follows immediately that the random variable $f(X)$ that minimizes $E[\big(Y-f(X)\big)^2]$ is $f(X) = E(Y|X)$.

## Exercise 1.2

### a)

Let $X = (X_1, X_2, \dots, X_n)$. Then

$$E[\big(X_{n+1} - f(X)\big)^2|X] = E[\big(X_{n+1} - E(X_{n+1}|X) + E(X_{n+1}|X) - f(X)\big)^2|X] =$$
$$E[\big(X_{n+1} - E(X_{n+1}|X)\big)^2|X] + 2E[\big(X_{n+1} - E(X_{n+1}|X)\big)\big(E(X_{n+1}|X) - f(X)\big)|X]$$
$$+ E[\big(E(X_{n+1}|X) - f(X)\big)^2|X] =$$
$$E[\big(X_{n+1} - E(X_{n+1}|X)\big)^2|X] + 2\big(E(X_{n+1}|X) - f(X)\big)E[\big(X_{n+1} - E(X_{n+1}|X)\big)|X]$$
$$+ E[\big(E(X_{n+1}|X) - f(X)\big)^2|X] =$$
$$E[\big(X_{n+1} - E(X_{n+1}|X)\big)^2|X] + E[\big(E(X_{n+1}|X) - f(X)\big)^2|X] \geq E[\big(X_{n+1} - E(X_{n+1}|X)\big)^2|X]$$

because $E(X_{n+1}|X)$ is a function of $X$ and $E(g(X)X_{n+1}|X) = g(X)E(X_{n+1}|X)$ for any function $g$ such that $E(g(X)X_{n+1})$ exists.

It follows that

$$E[\big(X_{n+1} - E(X_{n+1}|X)\big)^2|X] \leq E[\big(X_{n+1} - f(X)\big)^2|X]$$

for any function $f$. Hence $E[\big(X_{n+1} - E(X_{n+1}|X)\big)^2|X]$ is minimized when $f(X) = E(X_{n+1}|X)$.

### b)

Since

$$E[\big(X_{n+1} - E(X_{n+1}|X)\big)^2] = E\big(E[\big(X_{n+1} - E(X_{n+1}|X)\big)^2|X]\big)$$
$$\leq E\big(E[\big(X_{n+1} - f(X)\big)^2|X]\big) = E[\big(X_{n+1} - f(X)\big)^2]$$

it follows immediately that the random variable $f(X)$ that minimizes $E[\big(X_{n+1} - f(X)\big)^2]$ is again $f(X) = E(X_{n+1}|X)$.

### c)

By b) the minimum mean-squared error predictor of $X_{n+1}$ in terms of $X = (X_1, X_2, \dots, X_n)$ when $X_t \sim IID(\mu, \sigma^2)$ is

$$E(X_{n+1}|X) = E(X_{n+1}) = \mu$$

### d)

Suppose that $\sum_{i=1}^n \alpha_i X_i$ is an unbiased estimator for $\mu$, that is, $\sum_{i=1}^n \alpha_i = 1$. Then

$$E[\big(\sum_{i=1}^n \alpha_i X_i - \mu\big)^2] = E[\big(\sum_{i=1}^n \alpha_i X_i - \overline{X}\big)^2] + 2E[\big(\sum_{i=1}^n \alpha_i X_i - \overline{X}\big)\big(\overline{X} - \mu\big)] + E[\big(\overline{X} - \mu\big)^2] \geq E[\big(\overline{X} - \mu\big)^2]$$

since the second term is zero: $E[\big(\sum_{i=1}^n \alpha_i X_i - \overline{X}\big)\big(\overline{X} - \mu\big)] = Cov(\sum_{i=1}^n \alpha_i X_i - \overline{X}, \overline{X}) = Cov(\sum_{i=1}^n \alpha_i X_i, \sum_{i=1}^n \frac{1}{n} X_i) - Cov(\sum_{i=1}^n \frac{1}{n} X_i, \sum_{i=1}^n \frac{1}{n} X_i) = \sum_{i=1}^n \frac{\alpha_i}{n}\sigma^2 - \sum_{i=1}^n \frac{1}{n^2}\sigma^2 = 0$.

**e)**

Again, suppose that $\sum_{i=1}^{n} \alpha_i X_i$ is an unbiased estimator for $\mu$, that is, $\sum_{i=1}^{n} \alpha_i = 1$. Then

$$E[\big(X_{n+1} - \sum_{i=1}^{n} \alpha_i X_i\big)^2] = E[\big(X_{n+1} - \overline{X}\big)^2] + 2E[\big(X_{n+1} - \overline{X}\big)\big(\overline{X} - \sum_{i=1}^{n} \alpha_i X_i\big)] + E[\big(\overline{X} - \sum_{i=1}^{n} \alpha_i X_i\big)^2]$$
$$\geq E[\big(X_{n+1} - \overline{X}\big)^2]$$

since the second term is zero: $Cov\big(X_{n+1} - \overline{X}, \overline{X} - \sum_{i=1}^{n} \alpha_i X_i\big) = -Cov(\overline{X}, \overline{X}) + Cov(\overline{X}, \sum_{i=1}^{n} \alpha_i X_i) = 0$ as in d).

**f)**

$$E\big(S_{n+1}|S_1, \ldots, S_n\big) = E\big(S_n + X_{n+1}|S_1, \ldots, S_n\big) = S_n + E\big(X_{n+1}|S_1, \ldots, S_n\big) = S_n + \mu$$

since $X_{n+1}$ is independent of $S_1, \ldots, S_n$.

## Exercise 1.3

**i)**

$E(X_t)$ is independent of $t$ since the distribution of $X_t$ is independent of $t$ and $E(X_t)$ exists.

**ii)**

Since $E[X_{t+h}X_t]^2 \leq E[X_{t+h}^2]E[X_t^2]$ for all integers $t, h$, and the joint distribution of $X_{t+h}$ and $X_t$ is independent of $t$, it follows that $E[X_{t+h}X_t]$ exists and is independent of $t$ for every integer $h$.

Combining i) and ii) it follows that $X_t$ is weakly stationary.

## Exercise 1.4

**a)**

$E(X_t) = a$ is independent of $t$.

$$Cov\big(X_{t+h}, X_t\big) = \begin{cases} (b^2 + c^2)\sigma^2 & ; \quad h = 0 \\ 0 & ; \quad h = \pm 1 \\ bc\sigma^2 & ; \quad h = \pm 2 \\ 0 & ; \quad |h| > 2 \end{cases}$$

which is independent of $t$. That is, $X_t$ is stationary.

**b)**

$E(X_t) = 0$ is independent of $t$.

$$Cov\big(X_{t+h}, X_t\big) = Cov\big(Z_1 \cos c(t+h) + Z_2 \sin c(t+h), Z_1 \cos ct + Z_2 \sin ct\big)$$
$$= \sigma^2\big(\cos c(t+h)\cos ct + \sin c(t+h)\sin ct\big) = \sigma^2 \cos ch$$

which is independent of $t$. That is, $X_t$ is stationary.

**c)**

$E(X_t) = 0$ is independent of $t$.

$$Cov\big(X_{t+1}, X_t\big) = \sigma^2 \cos c(t+1) \sin ct$$

which is not independent of $t$. That is, $X_t$ is not stationary (except in the special case when $c$ is an integer multiple of $2\pi$).

**d)**

$E(X_t) = a$ is independent of $t$.

$$Cov\big(X_{t+h}, X_t\big) = b^2\sigma^2$$

which is independent of $t$. That is, $X_t$ is stationary.

**e)**

$E(X_t) = 0$ is independent of $t$.

$$Cov\big(X_{t+h}, X_t\big) = \sigma^2 \cos c(t+h) \cos ct$$

which is not independent of $t$. That is, $X_t$ is not stationary (except in the special case when $c$ is an integer multiple of $2\pi$).

**f)**

$E(X_t) = 0$ is independent of $t$.

$$Cov\big(X_{t+h}, X_t\big) = E[X_{t+h}X_t] = E[Z_{t+h}Z_{t+h-1}Z_tZ_{t-1}] = \begin{cases} \sigma^4 & ; \quad h = 0 \\ 0 & ; \quad |h| > 0 \end{cases}$$

which is independent of $t$. That is, $X_t$ is stationary, and it is seen that in fact $X_t \sim WN(0, \sigma^4)$.

## Exercise 1.5

**a)**

The autocovariance function

$$\gamma_X(h) = \begin{cases} 1+\theta^2 & ; \quad h = 0 \\ \theta & ; \quad h = \pm 2 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

The autocorrelation function

$$\rho_X(h) = \begin{cases} 1 & ; \quad h = 0 \\ \frac{\theta}{1+\theta^2} & ; \quad h = \pm 2 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

For $\theta = 0.8$ it is obtained that

$$\gamma_X(h) = \begin{cases} 1.64 & ; \quad h = 0 \\ 0.8 & ; \quad h = \pm 2 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

$$\rho_X(h) = \begin{cases} 1 & ; \quad h = 0 \\ 0.488 & ; \quad h = \pm 2 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

**b)**
Let $\overline{X}_4 = \frac{1}{4}(X_1 + \ldots + X_4)$. Then

$$Var(\overline{X}_4) = Cov(\overline{X}_4, \overline{X}_4) = \frac{1}{16}\sum_{i=1}^{4}\sum_{i=1}^{4} Cov(X_i, X_j)$$
$$= \frac{1}{4}\big(\gamma_X(0) + \gamma_X(2)\big) = \frac{1}{4}\big(1.64 + 0.8\big) = 0.61$$

**c)**

$$Var(\overline{X}_4) = Cov(\overline{X}_4, \overline{X}_4) = \frac{1}{4}\big(\gamma_X(0) + \gamma_X(2)\big) = \frac{1}{4}\big(1.64 - 0.8\big) = 0.21$$

The negative lag 2 correlation in c) means that positive deviations of $X_t$ from zero tend to be followed two time units later by a compensating negative deviation, resulting in smaller variability in the sample mean than in b) (and also smaller than if the time series $X_t$ were IID(0, 1.64) in which case $Var(\overline{X}_4) = 0.41$).

## Exercise 1.15

**a)**

Since $s_t$ has period 12

$$\nabla_{12}X_t = \nabla_{12}(a + bt + s_t + Y_t) = 12b + Y_t - Y_{t-12}$$

so that

$$W_t := \nabla\nabla_{12}X_t = Y_t - Y_{t-1} - Y_{t-12} - Y_{t-13}.$$

Then $\mathrm{E}[W_t] = 0$ and

$$\begin{aligned}\mathrm{Cov}[W_{t+h}, W_t] &= \mathrm{Cov}[Y_{t+h} - Y_{t+h-1} - Y_{t+h-12} - Y_{t+h-13}, Y_t - Y_{t-1} - Y_{t-12} - Y_{t-13}] \\ &= 4\gamma(h) - 2\gamma(h-1) - 2\gamma(h+1) + \gamma(h-11) + \gamma(h+11) - 2\gamma(h-12) \\ &- 2\gamma(h+12) + \gamma(h+13) + \gamma(h-13)\end{aligned}$$

where $\gamma(\cdot)$ is the ACVF of $Y_t$. Since $\mathrm{E}[W_t]$ and $\mathrm{Cov}[W_{t+h}, W_t]$ are independent of $t$, $W_t$ is stationary. Also note that $\nabla_{12}X_t$ is stationary.

**b)**

Using $X_t = (a + bt)s_t + Y_t$ it is obtained that

$$\nabla_{12}X_t = bts_t - b(t-12)s_{t-12} + Y_t - Y_{t-12} = 12bs_{t-12} + Y_t - Y_{t-12}.$$

Now let $U_t = \nabla_{12}^2 X_t = Y_t - 2Y_{t-12} + Y_{t-24}$. Then $\mathrm{E}[U_t] = 0$ and

$$\begin{aligned}\mathrm{Cov}[U_{t+h}, U_t] &= \mathrm{Cov}[Y_{t+h} - 2Y_{t+h-12} + Y_{t+h-24}, Y_t - 2Y_{t-12} + Y_{t-24}] \\ &= 6\gamma(h) - 4\gamma(h+12) - 4\gamma(h-12) + \gamma(h+2) + \gamma(h-24),\end{aligned}$$

which is independent of $t$. Hence $U_t$ is stationary.