Norwegian University of Science and Technology
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Solutions Exercise 3, 2008

Exercise 2.18

Given the MA(1) process

$$X_t = Z_t - \theta Z_{t-1}$$

where $|\theta| < 1$, and $Z_t \sim WN(0, \sigma^2)$. Represented as an AR(∞) process, it assumes the form

$$Z_t = X_t + \theta X_{t-1} + \theta^2 X_{t-2} + \dots$$

Setting $t = n + 1$ in the last equation and applying $\tilde{P}_n$ to each side, leads to the result

$$\tilde{P}_n X_{n+1} = -\sum_{j=1}^{\infty} \theta^j X_{n+1-j} = -\theta Z_n$$

Prediction error = $X_{n+1} - \tilde{P}_n X_{n+1} = Z_{n+1}$. Hence, MSE = $E[Z_{n+1}^2] = \sigma^2$.

Exercise 2.19

The given MA(1)-model is

$$X_t = Z_t - Z_{t-1}; \quad t \in \mathbb{Z}$$

where $Z_t \sim \mathrm{WN}(0, \sigma^2)$.

The vector $\mathbf{a} = (a_1, \dots, a_n)'$ of the coefficients that provide the best linear predictor (BLP) of X_{n+1} in terms of $\mathbf{X} = (X_n, \dots, X_1)'$ satisfies the equation

$$\Gamma_n \mathbf{a} = \gamma_{\mathbf{n}}$$

where the covariance matrix $\Gamma_n = Cov(\mathbf{X}, \mathbf{X})$ and $\gamma_{\mathbf{n}} = Cov(X_{n+1}, \mathbf{X}) = (\gamma(1), \dots, \gamma(n))'$. Since $\gamma(0) = 2\sigma^2$, $\gamma(1) = -\sigma^2$, $\gamma(h) = 0$ for $|h| > 1$, it follows that

$$\Gamma_n = \sigma^2 \begin{pmatrix} 2 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 2 \end{pmatrix}$$

and $\gamma_{\mathbf{n}} = \sigma^2(-1, 0, \dots, 0)'$. It can be shown, e.g. by induction, that the equations to be solved can be rewritten as follows

$$\begin{pmatrix} 2 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 3 & 2 & 0 & \dots & 0 & 0 \\ 0 & 0 & 4 & 3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & n & (n-1) \\ 0 & 0 & 0 & 0 & \dots & 1 & 2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \\ a_{n-1} \\ a_n \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \\ \vdots \\ (-1)^{n-1} \\ 0 \end{pmatrix}$$

The solution is found to be given as follows

$$a_j = (-1)^j \frac{n+1-j}{n+1}$$

Hence it is obtained that

$$P_n X_{n+1} = \sum_{j=1}^{n} (-1)^j \frac{n+1-j}{n+1} X_{n+1-j}$$

The mean square error is

$$E[(X_{n+1} - P_n X_{n+1})^2] = \gamma(0) - \mathbf{a}'\gamma_{\mathbf{n}} = 2\sigma^2 + a_1\sigma^2 = \sigma^2\Big(1 + \frac{1}{n+1}\Big)$$

Exercise 2.20

We have to prove that

$$Cov(X_n - \hat{X}_n, X_j) = E[(X_n - \hat{X}_n)X_j] = 0$$

for $j = 1, \dots, n-1$. This follows from equations (2.5.5) for suitable values of n and h with $a_0 = 0$ (since we may assume that $E[X_n] = 0$). This clearly implies that

$$E[(X_n - \hat{X}_n)(X_k - \hat{X}_k)] = 0$$

for $k = 1, \dots, n-1$, since $\hat{X}_k$ is a linear combination of $X_1, \dots, X_{k-1}$.

Exercise 2.21

In this exercise we shall determine the best linear predictor (BLP) $P(X_3|\mathbf{W}_\alpha)$ wrt three different vector variables $\mathbf{W}_\alpha$, $\alpha = a, b, c$. Let $\Gamma_\alpha = Cov(\mathbf{W}_\alpha, \mathbf{W}_\alpha)$ and $\gamma_\alpha = Cov(X_3, \mathbf{W}_\alpha)$.

The given MA(1)-model is

$$X_t = Z_t + \theta Z_{t-1}; \quad t \in \mathbb{Z}$$

where $Z_t \sim \mathrm{WN}(0, \sigma^2)$.

a)

In this case we have $\mathbf{W}_a = (W_1, W_2)' = (X_2, X_1)'$. Hence

$$\Gamma_a = \sigma^2 \begin{pmatrix} 1+\theta^2 & \theta \\ \theta & 1+\theta^2 \end{pmatrix}$$

and $\gamma_a = Cov(X_3, \mathbf{W}_a) = (\gamma(1), \gamma(2))' = \sigma^2(\theta, 0)$. The equation $\Gamma_a(a_1, a_2)' = \gamma_a$, or

$$(1+\theta^2)a_1 + \theta a_2 = \theta$$

$$\theta a_1 + (1+\theta^2)a_2 = 0$$

has the solution

$$a_1 = \frac{\theta(1+\theta^2)}{(1+\theta^2)^2 - \theta^2} \qquad a_2 = \frac{-\theta^2}{(1+\theta^2)^2 - \theta^2}$$

We obtain the BLP

$$P(X_3|X_2, X_1) = \frac{\theta}{(1+\theta^2)^2 - \theta^2}\left((1+\theta^2)X_2 - \theta X_1\right)$$

The mean square error

$$\begin{aligned} E[(X_3 - P(X_3|X_2, X_1))^2] &= \mathrm{Var}(X_3) - (a_1, a_2)\gamma_a = \sigma^2(1+\theta^2) - a_1\sigma^2\theta \\ &= \sigma^2(1+\theta^2)\left(1 - \frac{\theta^2}{(1+\theta^2)^2 - \theta^2}\right) \end{aligned}$$

b)

Here $\mathbf{W}_b = (W_1, W_2)' = (X_4, X_5)'$. With this choice, it follows that $\Gamma_b = \Gamma_a$, and $\gamma_b = \gamma_a$. It follows immediately that the BLP is given by

$$P(X_3|X_4, X_5) = \frac{\theta}{(1+\theta^2)^2 - \theta^2}\left((1+\theta^2)X_4 - \theta X_5\right)$$

And the mean square error is the same as in a)

$$E[(X_3 - P(X_3|X_4, X_5))^2] = \sigma^2(1+\theta^2)\left(1 - \frac{\theta^2}{(1+\theta^2)^2 - \theta^2}\right)$$

c)

Now, $\mathbf{W}_b = (W_1, W_2, W_3, W_4)' = (X_2, X_1, X_4, X_5)'$. It then follows that

$$\Gamma_c = \sigma^2 \begin{pmatrix} \Gamma_a & \bar{0} \\ \bar{0} & \Gamma_a \end{pmatrix}$$

where $\bar{0}$ denotes a 2×2 zero-matrix. Also, $\gamma_c = (\gamma_a', \gamma_a')'$. Hence, it follows that the solution to the equation $\Gamma_c(a_1, \ldots, a_4)' = \gamma_c$ is given by $a_3 = a_1$ and $a_4 = a_2$, where a_1 and a_2 are as given in a) or b). The BLP is therefore

$$P(X_3|X_2, X_1, X_4, X_5) = \frac{\theta}{(1+\theta^2)^2 - \theta^2}\left((1+\theta^2)[X_2 + X_4] - \theta[X_1 + X_5]\right)$$

with mean square error

$$E[(X_3 - P(X_3|X_2, X_1, X_4, X_5))^2] = Var(X_3) - (a_1, a_2, a_3, a_4)\gamma_c = \sigma^2(1+\theta^2) - 2a_1\sigma^2\theta$$
$$= \sigma^2(1+\theta^2)\left(1 - \frac{2\theta^2}{(1+\theta^2)^2 - \theta^2}\right)$$

d)

See above.

Exercise 2.22

We shall determine the best linear predictor (BLP) $P(X_3|\mathbf{W}_\alpha)$ wrt three different vector variables $\mathbf{W}_\alpha$, $\alpha = a, b, c$. Let $\Gamma_\alpha = Cov(\mathbf{W}_\alpha, \mathbf{W}_\alpha)$ and $\gamma_\alpha = Cov(X_3, \mathbf{W}_\alpha)$.

The given causal (stationary) AR(1)-model is

$$X_t = \phi X_{t-1} + Z_t; \quad t \in \mathbb{Z}$$

where $Z_t \sim \mathrm{WN}(0, \sigma^2)$. Causality implies that $|\phi| < 1$. Hence, the ACVF $\gamma(h) = \sigma^2(1-\phi^2)^{-1}\phi^{|h|}$.

a)

In this case we have $\mathbf{W}_a = (W_1, W_2)' = (X_2, X_1)'$. Hence

$$\Gamma_a = \frac{\sigma^2}{1-\phi^2}\begin{pmatrix} 1 & \phi \\ \phi & 1 \end{pmatrix}$$

and $\gamma_a = Cov(X_3, \mathbf{W}_a) = (\gamma(1), \gamma(2))' = \frac{\sigma^2}{1-\phi^2}(\phi, \phi^2)'$. The equation $\Gamma_a(a_1, a_2)' = \gamma_a$, or

$$a_1 + \phi a_2 = \phi$$
$$\phi a_1 + a_2 = \phi^2$$

has the solution

$$a_1 = \phi \qquad a_2 = 0$$

We obtain the BLP

$$P(X_3|X_2, X_1) = \phi X_2$$

The mean square error

$$E[(X_3 - P(X_3|X_2, X_1))^2] = \mathrm{Var}(X_3) - (a_1, a_2)\gamma_a = \frac{\sigma^2}{1-\phi^2} - \frac{\sigma^2\phi^2}{1-\phi^2} = \sigma^2$$

b)

Here $\mathbf{W}_b = (W_1, W_2)' = (X_4, X_5)'$. With this choice, it follows that $\Gamma_b = \Gamma_a$, and $\gamma_b = \gamma_a$. It follows immediately that the BLP is given by

$$P(X_3|X_4, X_5) = \phi X_4$$

And the mean square error is

$$E[(X_3 - P(X_3|X_4, X_5))^2] = Var(X_3) - (a_1, a_2)\gamma_b = \frac{\sigma^2}{1-\phi^2} - \frac{\sigma^2\phi^2}{1-\phi^2} = \sigma^2$$

c)

Now, $\mathbf{W}_c = (W_1, W_2, W_3, W_4)' = (X_2, X_1, X_4, X_5)'$. It then follows that

$$\Gamma_c = \frac{\sigma^2}{1-\phi^2}\begin{pmatrix} 1 & \phi & \phi^2 & \phi^3 \\ \phi & 1 & \phi^3 & \phi^4 \\ \phi^2 & \phi^3 & 1 & \phi \\ \phi^3 & \phi^4 & \phi & 1 \end{pmatrix}$$

where $\gamma_c = (\gamma_a', \gamma_a')'$. Hence, the following set of equations is obtained

$$a_1 + \phi a_2 + \phi^2 a_3 + \phi^3 a_4 = \phi$$

$$\phi a_1 + a_2 + \phi^3 a_3 + \phi^4 a_4 = \phi^2$$

$$\phi^2 a_1 + \phi^3 a_2 + a_3 + \phi a_4 = \phi$$

$$\phi^3 a_1 + \phi^4 a_2 + \phi a_3 + a_4 = \phi^2$$

It is seen that the first two equations give $a_2 = 0$, while the last two equations give $a_4 = 0$. Then it is found that

$$a_1 = a_3 = \frac{\phi}{1+\phi^2}$$

The BLP is therefore

$$P(X_3|X_2, X_1, X_4, X_5) = \frac{\phi}{1+\phi^2}[X_2 + X_4]$$

with mean square error

$$E[(X_3 - P(X_3|X_2, X_1, X_4, X_5))^2] = Var(X_3) - (a_1, a_2, a_3, a_4)\gamma_c = \frac{\sigma^2}{1-\phi^2} - \frac{\sigma^2}{1-\phi^2}\frac{2\phi^2}{1+\phi^2}$$
$$= \frac{\sigma^2}{1+\phi^2}$$

d)

See above.