# Pricing of Asian Options by Numerical Path Integration

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#### Abstract

The paper presents a novel approach to the use of path integration for calculating the price of options. In particular, the case of Asian options is considered. The first step is to write down an Itô stochastic differential equation for a 2D state space vector process describing the temporal development of the instantaneous value of the underlying asset and its time integral. This equation is solved numerically by applying a special backward-forward time stepping procedure combined with splines interpolation. This makes it possible to achieve a very high degree of accuracy in the calculations, which is illustrated by several example results.

#### 1 Introduction

A financial derivative or an option is a contract whose value is determined by an underlying instrument. Typical underlyings are stocks and bonds. Such contracts are often used in the risk management of investment portfolios. The global derivative market is larger than the gross domestic product of most industrialized nations and is thus of fundamental importance in the world's financial system. The recent years has seen a significantly increased activity in the exotic derivative market. Options not traded on an exchange may have more complicated rules that the usual put and call options written on the price of a stock. Typical examples of exotic derivatives are options written on stock price averages over a specified period rather than on the price at the end of the period. These are also known as Asian options, and they have attracted a lot of attention over the years, especially in financial mathematical circles. Recently, some exact solutions have been found [1, 2], which makes it possible to check and verify approximate numerical procedures. In the following, we shall develop a numerical method to price Asian options, and our results will be compared with the corresponding exact results.

## 2 Model

It is assumed that the price process of a financial asset obeys the following dynamic model given by the Itô stochastic differential equation (SDE)

$$dS(t) = \mu[S(t)]dt + \sigma[S(t)]dB(t), \tag{1}$$

where  $S(0) = s_0$  is a specified positive constant.  $\mu(\cdot)$  and  $\sigma(\cdot)$  are positive functions satisfying suitable regularity conditions, cf. [3]. B(t) denotes a standard Brownian motion. Now define the running average process as

$$Y(t) = \frac{1}{t} \int_0^t S(u) du.$$
<sup>(2)</sup>

For the version of an Asian call option adopted in this paper, its value with maturity time T and strike X in a market with riskless interest rate r is then equal to the discounted expected payoff:

$$p = e^{-rT} E[(Y(T) - X)^+].$$
(3)

where  $(x)^+ = \max(x, 0)$ . For the numerical analysis, we shall replace Y(t) by the integral of the price process, that is

$$Z(t) = \int_0^t S(u)du \tag{4}$$

For a fixed strike Asian option, X is a constant, X = K say. If the probability density function (PDF)  $f_{Z(T)}(\cdot)$  of Z(T) is known, then clearly

$$p = e^{-rT} \frac{1}{T} \int_{TK}^{\infty} (v - TK) f_{Z(T)}(v) dv.$$
 (5)

In the case of a floating strike Asian option, X = S(T). Hence, if the joint PDF  $f_{S(T)Z(T)}(\cdot, \cdot)$  of S(T) and Z(T) is known, the price of the call option can be calculated as follows

$$p = e^{-rT} \frac{1}{T} \int_0^\infty \int_{T_u}^\infty (v - Tu) f_{S(T) Z(T)}(u, v) \, dv \, du.$$
(6)

Both types of Asian options can be priced by using the method developed in this paper. The reason for this is that our method is based on obtaining a numerical approximation to the option's value by calculating the temporal development of the joint PDF  $f_{S(t)Z(t)}(s, z)$  of the price process and its integral over the time interval (0, T). Our vehicle is the SDE of the state space vector process (S(t), Z(t))', that is

$$dS(t) = \mu[S(t)] dt + \sigma[S(t)] dB(t)$$
(7)

$$dZ(t) = S(t)dt \tag{8}$$

with initial conditions  $S(0) = s_0$  and Z(0) = 0 using numerical path integration, see e.g. [4, 5], a method that rarely appears in the financial literature. For a thorough explanation of how Feynman's original path integral concept from quantum physics can be adapted to solve problems in financial mathematics, the reader may consult [6]. A direct implementation of these ideas into the construction of a numerical algorithm is detailed in [7], but no numerical results are given. Consequently, it is hard to assess the efficiency and accuracy of the proposed algorithm.

In the present paper we have chosen a different and more direct approach to the use of the path integration idea in financial engineering, specifically to the pricing of Asian options. Our approach also differs significantly in the way the numerical calculations are carried out. By introducing a backwardforward time stepping procedure combined with spline interpolation, it is possible to achieve a high degree of numerical accuracy in the calculations. It should be noted, however, that the final aim of this research is not primarily to price Asian options. For some standard models they can already be reliably priced by existing methods. The aim is rather to suggest a more general method that may be applied to a wide range of situations, where there may be no other alternative to Monte Carlo simulations. Verifying that the method works on Asian options for which there already exist some exact results, is a step in testing and verifying the method.

## **3** Numerical Path Integration

Systems of SDEs like (7) and (8) can be simulated by an arsenal of numerical methods of varying accuracy [8]. The simplest method is the so-called Euler-Maruyama method, which is the stochastic generalization of the well-known Euler method for ordinary differential equations. Let

$$0 = t_0 < t_1 < \ldots < t_N = T$$

be a partition of the time interval (0, T), let  $S_i$  denote  $S(t_i)$  and  $Z_i$  denote  $Z(t_i)$ . The Euler-Maruyama method then reads

$$S_{i+1} = S_i + \mu[S_i] \,\Delta t_i + \sigma[S_i] \,\Delta B_i \tag{9}$$

$$Z_{i+1} = Z_i + S_i \Delta t_i \tag{10}$$

where  $\Delta t_i = t_{i+1} - t_i$  and  $\Delta B_i = B(t_{i+1}) - B(t_i)$ . The transition probability density (TPD) associated with the Markov chain  $\{S_i\}_{i=0}^{\infty}$  as given by equation (9), is denoted by  $p(s, t_{i+1} | s', t_i)$ , which is the conditional density of  $S(t_{i+1})$  at s given  $S(t_i) = s'$ . Since  $\Delta B_i$  is normally distributed with expectation value zero and variance equal to  $\Delta t_i$ , equations (9) and (10) implicitly define a transition probability distribution for the two-dimensional Markov chain  $\{(S_i, Z_i)'\}_{i=0}^{\infty}$  as follows

$$P(S_{i+1} < a, Z_{i+1} < b | S_i = s_i, Z_i = z_i)$$
  
=  $\int_{-\infty}^{a} \int_{-\infty}^{b} d\delta_{z_i + s_i \Delta t_i}(z) p(s, t_{i+1} | s_i, t_i) ds$   
=  $\int_{-\infty}^{a} \int_{-\infty}^{b} d\delta_{z_i + s_i \Delta t_i}(z) \phi_N(s; \mu_i, \sigma_i^2) ds$  (11)

where  $\mu_i = s_i + \mu(s_i) \Delta t_i$ ,  $\sigma_i^2 = \sigma(s_i)^2 \Delta t_i$ , and  $\phi_N(s; \mu_i, \sigma_i^2)$  denotes the PDF of a normal variate with mean value  $\mu_i$  and variance  $\sigma_i^2$ , that is

$$\phi_N(s;\mu_i,\sigma_i^2) = \frac{1}{\sqrt{2\pi\sigma_i}} \exp\left(-\frac{(s-\mu_i)^2}{2\sigma_i^2}\right).$$
(12)

and  $\delta_c$  denotes the Dirac measure located at c, that is,  $\delta_c(A) = 1$  if  $c \in A$  for any set A, and  $\delta_c(A) = 0$  if  $c \notin A$ .

Assuming that the random variables  $S_i$  and  $Z_i$  are specified by a joint PDF  $f_i$ , the total probability law then implies that the joint probability distribution function of  $S_{i+1}$  and  $Z_{i+1}$  is given as follows

$$P(S_{i+1} < a, Z_{i+1} < b)$$
  
=  $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(S_{i+1} < a, Z_{i+1} < b \mid S_i = s', Z_i = z') f_i(s', z') dz' ds'$  (13)

From this relation and equation (11), it can now be shown that the random variables  $S_{i+1}$  and  $Z_{i+1}$  are also characterized by a PDF,  $f_{i+1}$ , which satisfies the equation

$$f_{i+1}(s,z) = \int_{-\infty}^{\infty} p(s,t_{i+1}|s',t_i) f_i(s',z-s'\Delta t) ds' = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma(s')^2 \Delta t_i}} \exp\left(-\frac{(s-s'-\mu(s') \Delta t_i)^2}{2\sigma(s')^2 \Delta t_i}\right) f_i(s',z-s'\Delta t_i) ds'$$
(14)

We know that S(t) and Z(t) both have positive values for t > 0 when S(0) > 0. A scrutiny of the Euler-Maruyama approximation given by equations (9) and (10) immediately reveals that this condition is violated since there is a positive probability of negative values for  $S_{i+1}$ . Of course, this probability is in general very small, or even negligible, for small values of  $\Delta t_i$ . Therefore, this approximation, if properly implemented, will give a good numerical representation of the dynamics of S(t) and Z(t).

There is yet a possibility to improve on equation (14). If we go back to equation (10), it is clear that an improved approximation is obtained if we write

$$Z_{i+1} = Z_i + S_i \Delta t_i + \frac{1}{2} \frac{S_{i+1} - S_i}{\Delta t_i} (\Delta t_i)^2 = Z_i + \frac{1}{2} (S_{i+1} + S_i) \Delta t_i$$
(15)

which is recognized as a trapezoidal integration formula. Equations (9) and (15) again provides us with a Markov chain  $\{(S_i, Z_i)'\}$ , and it is obtained that

$$f_{i+1}(s,z) = \int_{-\infty}^{\infty} p(s,t_{i+1}|s',t_i) f_i(s',z-\frac{1}{2}(s+s')\Delta t_i) ds'$$
(16)

With given numerical initial conditions  $S(0) = s_0 > 0$  and Z(0) = 0, the initial joint PDF is not a proper PDF but rather a product of delta distributions, that is, of singular type. Hence we may write formally

$$f_0(s, z) = \delta(s - s_0) \,\,\delta(z).$$
 (17)

However, this kind of initial distribution is not convenient numerically. Instead, a proper initial PDF to use in equation (14) or (16) is sought. An immediate idea is then to try to derive a closed form expression for the first nonsingular PDF  $f_i$  by analytical calculation. Now,  $f_1$  is still of singular type for both equations, while  $f_2$  is nonsingular. However, in the general case it is not possible to derive an explicit expression for  $f_2$  without making approximations. If these approximations are too heavy handed, which is hard to avoid, unwanted loss of accuracy have been observed to occur. Instead, the chosen approach is to make  $f_1$  slightly nonsingular in the following manner. For the case of equation (16), the random variable  $\tilde{Z}_1 = Z_1 + E = (S_1 + s_0) \Delta t/2 + E$ is introduced, where  $E \sim N(0, e^2)$  and chosen to be independent of  $S_1$ . Hence, by choosing e sufficiently small,  $f_1(s, z) \approx \tilde{f}_1(s, z) = f_{S_1\tilde{Z}_1}(s, z) =$  $f_{\tilde{Z}_1|S_1}(z|s) f_{S_1}(s) = \phi_N(z; (s+s_0) \Delta t/2, e^2) \cdot \phi_N(s; s_0 + \mu(s_0) \Delta t, \sigma(s_0)^2 \Delta t)$ .

Using equations (14) or (16) recursively for i = 1, 2, 3, ..., an approximation is obtained for the time-dependent joint PDF associated with the vector stochastic process (S(t), Z(t)). The marginal density function used to value the option in equation (6) is integrated out of the joint density.

So far the analysis has been based on the Euler-Maruyama approximation to equation (7), that is, on equation (9), which is tantamount to the approximation  $\int_{t_i}^{t_{i+1}} \sigma[S(t)] dB(t) = \sigma[S_i] (B_{i+1} - B_i) = \sigma[S_i] \Delta B_i$ . The advantage of this approximation is obvious from the preceding analysis, viz. that the TPD  $p(s, t_{i+1} | s', t_i)$  can be represented by a Gaussian density. As discussed extensively in [8], there are several ways of improving on the simple Euler-Maruyama approximation, both by weak and strong discretization schemes. Since the goal of our work is to calculate the probability law of the state space vector process, we may limit ourselves to the weak schemes. In particular, we shall consider the simplified weak Taylor scheme of order 2.0. The Euler-Maruyama scheme is of weak order 1.0. According to [8], the simplified weak order 2.0 Taylor scheme for the conditional random variable  $\tilde{S}_{i+1} = \{S_{i+1} | S_i = s_i\}$  may be written as

$$\tilde{S}_{i+1} = \alpha_i + \beta_i \,\Delta B_i + \gamma_i \,\Delta B_i^2 \tag{18}$$

Here

$$\alpha_{i} = s_{i} + \mu(s_{i}) \Delta t_{i} - \frac{1}{2} \sigma(s_{i}) \sigma'(s_{i}) \Delta t_{i} + \frac{1}{2} \left[ \mu(s_{i}) \, \mu'(s_{i}) + \frac{1}{2} \mu''(s_{i}) \, \sigma(s_{i})^{2} \right] \Delta t_{i}^{2}$$
(19)

$$\beta_i = \sigma(s_i) + \frac{1}{2} \left[ \mu'(s_i) \,\sigma(s_i) + \,\mu(s_i) \,\sigma'(s_i) + \frac{1}{2} \sigma''(s_i) \,\sigma(s_i)^2 \,\right] \Delta t_i \tag{20}$$

and

$$\gamma_i = \frac{1}{2} \,\sigma(s_i) \,\sigma'(s_i) \tag{21}$$

The prime ' denotes derivation with respect to the argument, that is,  $\mu'(s) = d\mu(s)/ds$ , and so on. Convergence of the present weak Taylor scheme of order 2.0 is guaranteed if the functions  $\mu(s)$  and  $\sigma(s)$  satisfy certain regularity conditions, cf. [8].

Having achieved the representation of equation (18), we may proceed to calculate  $p(s, t_{i+1} | s', t_i)$ . This TPD can, of course, still be expressed in closed form since  $\tilde{S}_{i+1}$  is a quadratic expression in the Gaussian variable  $\Delta B_i$ . Let  $\xi_i^{\pm}$  denote the two solutions of the equation

$$s = h(\xi) = \alpha_i + \beta_i \,\xi + \gamma_i \,\xi^2 \tag{22}$$

That is

$$\xi_i^{\pm} = -\frac{\beta_i}{2\gamma_i} \pm \sqrt{\frac{s - \alpha_i}{\gamma_i} + \left(\frac{\beta_i}{2\gamma_i}\right)^2} \tag{23}$$

It is then obtained that

$$p(s, t_{i+1} | s_i, t_i) = \sum_{\varepsilon = +, -} \frac{\phi_N(\xi_i^{\varepsilon}; 0, \Delta t)}{|h'(\xi_i^{\varepsilon})|} \\ = \left(\frac{s - \alpha_i}{\gamma_i} + \left(\frac{\beta_i}{2\gamma_i}\right)^2\right)^{-1/2} \left[\phi_N(\xi_i^+; 0, \Delta t) + \phi_N(\xi_i^-; 0, \Delta t)\right]$$
(24)

for  $\frac{s-\alpha_i}{\gamma_i} + \left(\frac{\beta_i}{2\gamma_i}\right)^2 > 0$ . So even if the TPD  $p(s, t_{i+1}|s_i, t_i)$  is more complicated for the weak order 2.0 approximation above than the previous TPD, which was simply a Gaussian density, it is still tractable for numerical calculations. It is therefore of interest to explore the impact of this approximation on the numerical accuracy of the calculated values for the option price by combining equations (16) and (24).

#### 4 Implementation

To calculate the temporal development of  $f_i(s, z)$ , we carry out the recursive integration of equation (14) numerically. The sequence of joint probability densities

$$f_1(s,z), f_2(s,z), f_3(s,z), \ldots$$

are represented by their numerical values on a  $200 \times 200$  grid which is updated at each time-step. We carry out 5000 Monte Carlo simulations of (S(t), Z(t)), keeping records of the highest and the lowest observed value at each time-step between 0 and T. The grid is delimited by 0.5 times the lowest and 1.5 times the highest values. Since the joint PDF develops from an initial density that is almost concentrated in a point in the (s, z)-plane into a smooth surface covering a larger area, such an adaptive grid contributes to keep the number of grid points relatively modest, thus saving CPU time.

When carrying out a numerical integration of equation (14) or (16) for each grid point, it is sufficient to limit the integration to an interval where the transition probability density has a substantially non-zero value. In practice, this is done as follows: For a given s, the Euler-Maruyama algorithm with a negative time-step will define a normal distribution as a function of s'. To include practically all the probability mass, the integration limits are set to the center of that distribution plus/minus six standard deviations, though not exceeding the grid boundaries. A detailed description of this backwardforward time stepping procedure, which is an essential part of the numerical implementation, is given in [5].

Simpson's method with 26 partitions has been used to carry out the integration. Since the quadrature algorithm in general calls for the PDF value at locations that are not grid points bicubic interpolation of  $f_i$  is performed. This is the most important factor in determining the density of grid points. On the one hand, to save CPU time, the density of grid points should be as low as possible. On the other hand, the density must be high enough to ensure that the interpolation provides sufficient accuracy for the integration to have the required precision.

Having divided the year into 360 time-steps, this procedure is repeated until the options maturity time T is reached. The marginal density function used to value the fixed strike option in equation (5) is integrated out of the joint density.

#### 5 Numerical Results

In this paper, we shall carry out calculations for the classical Black and Scholes model for the dynamics of the underlying asset. That is, equation (1) is rewritten in the form

$$dS(t) = r S(t) dt + \sigma S(t) dB(t), \qquad (25)$$

where r and  $\sigma$  are positive constants. For this model the order 2.0 weak Taylor approximation has  $\alpha_i = s_i \left[1 + (r - \sigma^2/2) \Delta t_i + r^2 \Delta t_i^2/2\right], \beta_i = \sigma s_i (1 + r \Delta t_i)$  and  $\gamma_i = \sigma^2 s_i/2$ .

It should be said, of course, that for the particular example of the Black and Scholes model, the TPD for S(t) is well known and can be expressed in closed form, viz. (s > 0, s' > 0)

$$p(s,t|s',t') = \frac{1}{\sqrt{2\pi\Delta t}\sigma s} \exp\left\{-\frac{\left(\ln s - \ln s' - (\mu - \frac{1}{2}\sigma^2)\Delta t\right)^2}{2\sigma^2\Delta t}\right\}$$
(26)

where  $\Delta t = t - t'$ , which is valid for any value of  $\Delta t > 0$ . Hence, there is no need for any approximation to p(s,t|s',t') in this case. However, since our focus is on developing a general approach, we have chosen not to use this exact, special case solution.

Lewis [2] considers a fixed strike Asian call option written on a stock not paying dividends, for which he gives exact option prices. In his examples, the underlying initial price  $S_0$  and strike K are equal to 2.0. The volatility is  $\sigma = 0.5$  and the risk-free interest rate is r = 0.05 and r = 0.20. The numerical path integration method described above was used to calculate the option price for various maturity times T in years. The obtained results are reported in the tables, where PI1 refers to the Euler-Maruyama approximation combined with equation (10), PI2 refers to the Euler-Maruyama approximation combined with equation (15), and PI3 refers to the order 2.0 weak Taylor approximation, also combined with equation (15). Calculations were carried out on a workstation, and one calculation took twenty seconds for PI2 when the maturity time is equal to a year. PI1 and PI2 calculations took about the same time to perform, while PI3 calculations took longer by a factor of about two.

The results in the tables indicate that there is little to gain by using PI3, that is, a 2. order weak Taylor scheme for the values of  $\Delta t$  needed to ensure high accuracy for the integrated process. It is also seen that the

| Т    | Exact    | PI1      | Error   | PI2      | Error   | PI3      | Error   |
|------|----------|----------|---------|----------|---------|----------|---------|
| 0.1  | 0.075067 | 0.073497 | -2.090% | 0.075096 | 0.039%  | 0.075083 | -0.021% |
| 0.25 | 0.120335 | 0.119339 | -0.831% | 0.120376 | 0.034%  | 0.120354 | -0.016% |
| 0.5  | 0.172296 | 0.171563 | -0.410% | 0.172295 | 0.015%  | 0.172264 | -0.003% |
| 1    | 0.246416 | 0.245874 | -0.220% | 0.246366 | -0.020% | 0.246319 | -0.039% |
| 2    | 0.350095 | 0.349436 | -0.188% | 0.349986 | -0.031% | 0.349909 | -0.053% |

Table 1: Risk-free interest rate r = 0.05

Table 2: Risk-free interest rate r = 0.20

| Т    | Exact    | PI1      | Error   | PI2      | Error   | PI3      | Error   |
|------|----------|----------|---------|----------|---------|----------|---------|
| 0.1  | 0.082117 | 0.080322 | -2.186% | 0.082021 | -0.005% | 0.082024 | -0.113% |
| 0.25 | 0.137038 | 0.135903 | -0.909% | 0.136998 | -0.029% | 0.137003 | -0.026% |
| 0.5  | 0.203184 | 0.202231 | -0.469% | 0.203229 | 0.022%  | 0.203240 | 0.028%  |
| 1    | 0.299968 | 0.299095 | -0.291% | 0.299831 | -0.046% | 0.299848 | -0.040% |
| 2    | 0.430616 | 0.429538 | -0.250% | 0.430351 | -0.062% | 0.430381 | -0.055% |

high accuracy is lost when PI1 is used, that is, when only the crude Euler approximation is used for the integrated process. Hence, for the subsequent calculations, only PI2 is used.

Rogers and Shi [9] present tables with calculated prices for fixed and floating strike Asian call options with a maturity period of one year. Their expectedly most accurate results are based on numerically solving a partial differential equation. Although the accuracy of their solutions is unknown, it appears to be good. It is therefore tempting to match our values with theirs. Results for the fixed strike option are listed in Table 3, while results for the floating strike option are shown in Table 4, where R&S refers to the results of Rogers and Shi.

Acceptable agreement is obtained for all combinations of interest rate and volatility except for r = 0.15 and  $\sigma = 0.1$ . The significant deviation observed was very hard to explain if the R&S value was correct, because

| r    | K   | R&S    | PI2    | Deviation |
|------|-----|--------|--------|-----------|
| 0.05 | 90  | 13.951 | 13.953 | 0.014%    |
| 0.05 | 100 | 7.944  | 7.948  | 0.050%    |
| 0.05 | 110 | 4.074  | 4.072  | -0.049%   |
| 0.09 | 90  | 14.981 | 14.987 | 0.040%    |
| 0.09 | 100 | 8.827  | 8.829  | 0.023%    |
| 0.09 | 110 | 4.698  | 4.696  | -0.043%   |
| 0.15 | 90  | 16.510 | 16.512 | 0.012%    |
| 0.15 | 100 | 10.208 | 10.209 | 0.010%    |
| 0.15 | 110 | 5.731  | 5.728  | -0.052%   |

Table 3: Fixed strike Asian option,  $\sigma = 0.30$ 

Table 4: Floating strike Asian option

| σ   | r    | R&S   | PI2   | Deviation |
|-----|------|-------|-------|-----------|
| 0.1 | 0.05 | 1.257 | 1.258 | 0.080%    |
| 0.1 | 0.09 | 0.709 | 0.709 | 0.000%    |
| 0.1 | 0.15 | 0.271 | 0.261 | -3.690%   |
| 0.2 | 0.05 | 3.401 | 3.405 | 0.118%    |
| 0.2 | 0.09 | 2.622 | 2.622 | 0.000%    |
| 0.2 | 0.15 | 1.723 | 1.710 | -0.754%   |
| 0.3 | 0.05 | 5.628 | 5.627 | -0.018%   |
| 0.3 | 0.09 | 4.736 | 4.740 | 0.084%    |
| 0.3 | 0.15 | 3.612 | 3.611 | -0.389%   |

a control calculation for the corresponding fixed strike option showed very good agreement. And since the two integrations we carry out for the two cases should give similar accuracy, it was decided to perform a massive MC calculation to check the correct value. On the basis of  $10^9$  MC simulations, the price was estimated to be 0.262. This gives an accuracy of the PI result equal to 0.382%, which is quite good. The reason for the discrepancy between this particular R&S result and the result from the MC simulations may be

explained by the discussions in [10]

Our best results are very similar in accuracy to results obtained by other numerical methods that are developed with the sole purpose of pricing Asian options. However, numerical path integration is more general. It has the advantage that it can be immediately applied to more complex situations, like dividend-paying stocks. Compared to Monte Carlo methods, it has the advantage of always coming up with the same result, and it spends less CPU time to arrive at a given accuracy.

#### 6 Conclusions and Suggestions

It has been demonstrated that the value of Asian options can be calculated fairly quickly and with a high degree of accuracy by numerical path integration. Considering that the results presented in this paper represent our first efforts to use numerical path integration to the problem of option pricing, it should indeed be expected that improvements in accuracy as well as computational speed can be achieved.

As already pointed out above, the most important aspect of numerical path integration is its flexibility. Generalization to large classes of more complex situations is straightforward. The method should be robust with respect to nonlinearity, time-dependence and the properties of the stochastic process, at least as long as the process driving the asset dynamics has independent increments. It is therefore the authors' intention to test the numerical path integration method on more complex models of financial instruments that cannot be easily valuated by traditional methods.

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